Lifestyle, Longevity, and Legacy Risks with Annuities in Retirement Portfolio Decumulation

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Abstract

Investors planning for retirement balance three "L"s: (1) lifestyle risk, hoping to maintain a consumption stream that provides a chosen standard of living, (2) longevity risk, hoping to remain solvent throughout their lifetime, and (3) legacy risk, hoping to leave a bequest to their heirs. We solve this multiple objective problem for a wide range of consumption and annuitization scenarios. For each scenario we apply dynamic programming to optimally evolve the investments in the non-annuitized portion of the portfolio so as to minimize longevity risk. Our dynamic programming approach has the advantages of (1) generating results that are far superior to what standard Monte Carlo methods, static portfolios, and target date fund glide paths can provide, and (2) not requiring utility functions, which are hard to specify for individuals. We show that investors who want to minimize their longevity and legacy risk and who are unable to annuitize their full consumption stream are best off avoiding even partial annuitization of their portfolio. For investors who are able to annuitize their full consumption stream, we quantify their longevity versus legacy risk trade-offs, enabling them to select the best annuity for their needs.

Highlights/Key Takeaways:

1. Reflecting a retiree's perspective, we consider the effect of choosing an annuity and choosing an annual consumption rate on the probability of remaining solvent during the retiree's lifetime and the expected bequest when they pass.
2. Dynamic programming is used to trade investments in the non-annuitized portion of the portfolio so as to optimize the retiree's solvency probability. Unlike traditional dynamic programming, probability, not utility, is optimized.
3. We find over a wide variety of annuity features that retirees increase their solvency probability the more they annuitize only if they are able to annuitize all their consumption needs. Otherwise, their solvency probability is optimized by eschewing annuities completely.

Keywords: annuities, retirement planning, longevity risk, bequest, dynamic programming

JEL Codes: G11, G41, G51

1 Introduction

> "Andre-Francois Raffray thought he had a great deal 30 years ago: He would pay a 90-year-old woman 2,500 francs (about $500) a month until she died, then move into her grand apartment in a town Vincent van Gogh once roamed.
>
> But this Christmas, Mr. Raffray died at age 77, having laid out the equivalent of more than $184,000 for an apartment he never got to live in.
>
> On the same day, Jeanne Calment, now listed in the Guinness Book of Records as the world's oldest person at 120, dined on foie gras, duck thighs, cheese and chocolate cake at her nursing home near the sought-after apartment in Arles, northwest of Marseilles in the south of France.
>
> She need not worry about losing income. Although the amount Mr. Raffray already paid is more than twice the apartment's current market value, his widow is obligated to keep sending that monthly check. If Mrs. Calment outlives her, too, then the Raffray children and grandchildren will have to pay.
>
> 'In life, one sometimes makes bad deals,' Mrs. Calment said on her birthday last Feb. 21."
>
> — The New York Times, December 29, 1995. (https://tinyurl.com/mj2x6z82)

Jeanne Calment passed away in Arles, on August 4, 1997, at the age of 122. While she was one of the oldest people who ever lived, the basic issues and concerns she had pertain to most retirees. Like Jeanne Calment, most retirees worry about outliving their savings, known in the financial planning community as "longevity risk."

There are reasons for retirees to be more concerned about longevity risk than before. Although most investors will live nowhere near as long as Jeanne Calment, recent advances in medicine have increased life expectancy. For example, in the United States, life expectancy has increased from 73.7 years in 1980 to 77.3 years in 2020[1]. Further, longevity risk has been exacerbated in the United States by a strong shift from employers away from defined benefit plans, like pensions, in favor of 401(k)s and Roth accounts (Department of Labor, 2019). This means that other than the small income Social Security provides, fewer Americans have guaranteed sources of income for life — like the Raffrays agreed to provide Jeanne Calment — even though they are living longer.

Therefore, it is increasingly important that financial planners, financial advisers, and insurance providers seek ways to hedge these newly heightened longevity risks. Annuities, especially as a replacement for the dwindling presence of pensions, can play an important role in this hedging process. This paper provides a dynamic programming framework for helping an investor choose the best annuity (amount, timing, and type) given a specified consumption rate during the investor's (uncertain) lifetime.

The dynamic programming approach we present reflects a different perspective from traditional dynamic programming approaches in that we use dynamic programming to evolve the non-annuitized investments in the investor's portfolio to maximize the probability of lifelong solvency,

[1] https://www.healthsystemtracker.org/chart-collection/u-s-life-expectancy-compare-countries/#item-life-expectancy-september-2021-update-chart-1. Comparable countries have higher life expectancies, with an increase from 74.5 years in 1980 to 82.1 years in 2020. Life expectancy peaked in 2019 at 78.9 years for the US and 82.6 years for comparable countries with the 2020 Covid pandemic unsurprisingly resulting in a downtick in longevity.

reflecting the same priority as the annuity. In contrast, traditional dynamic programming approaches look to maximize either the sum of the expected annual utility of consumption during the retiree's lifetime or the expected utility of the bequest, or a weighted combination of the two. In our perspective, the investor is more likely to be concerned about maximizing their probability of remaining solvent throughout their lifetime; that is, minimizing their longevity risk. This is more likely to reflect the priorities of an investor, especially one who is considering annuities, in that they are generally more concerned with staying solvent while maintaining a reasonable standard of living than they are with spending every dollar they can during their lifetime or prioritizing their heirs' bequest over their own needs.

This leads to the central questions of our paper: Given that the non-annuitized portion of the investor's portfolio is optimized to keep the investor solvent, is the investor more likely to stay solvent throughout their lifetime if (a) they put all of their portfolio into an annuity, (b) they put none of their portfolio into an annuity, or (c) they put a specific fraction of their portfolio into an annuity? And if the investor does put some or all of their portfolio into an annuity, how much legacy risk does the annuity generate? — or more specifically, what *quantitative* decrease does the annuity induce in the expected present value of the investor's bequest?

A fortunate byproduct of our prioritizing solvency in the non-annuitized portion of the portfolio is the considerable advantage of no longer requiring that utility functions be specified for annual consumption nor for bequests, which is notoriously difficult to accurately determine for individual investors. Nor do we need to make decisions for the investor regarding how to weigh the utility of consumption during the retiree's lifetime versus the utility of their bequest. Instead of maximizing the sum of the expected utility of annual consumption, we address lifestyle risk, that is, the concern of choosing the wrong annual consumption level, by having the investor select from an array of annual consumption levels that are constant over time in real dollars throughout the investor's lifetime. The investor can also select from a range of possible annuity choices to partially address these consumption needs. We then dynamically optimize the non-annuitized portion of the investor's wealth so as to minimize longevity risk. This generates a range of corresponding outcomes depicting (a) the (minimized) longevity risk by showing the (maximized) probability of remaining solvent throughout the (unknown) lifetime of the investor and (b) the legacy risk, which is measured by the expected bequest to the investor's heirs in present valued dollars.

The main results of the paper are:

1. Our approach allows investors a method of balancing lifestyle, longevity, and legacy risks that better reflects the way most retirees approach this balancing act. Specifically, it allows them to consider choosing a lifestyle (annual consumption) and an annuity, and then seeing the effect on the minimized longevity risk and the corresponding legacy risk. This enables investors to choose both a lifestyle and an annuity that best fits their longevity and legacy risk preferences.

2. By using dynamic programming, we are able to take advantage of optimal management of the non-annuitized portion of the retiree's wealth, rather than using Monte Carlo simulation over predetermined portfolio strategies, which is guaranteed not to produce a better result than our approach. Further, because we focus on minimizing longevity risk, our results are not dependent on significant and difficult utility function assumptions inherent in the traditional dynamic programming approaches that seek to maximize utility.

3. The results of our approach and analysis suggest that annuities are less desirable for a large class of investors than most utility maximization approaches suggest. Specifically,

for investors who do not have enough money to purchase an annuity that will cover all their consumption needs, our results show that completely eschewing annuities is actually the best strategy. In these cases, the presence of annuities not only reduces the expected bequest, which is to be expected, but also reduces the investor's solvency probability. This result holds under the wide variety of annuities we consider, including changing the inflation adjustment of the annuity payouts, changing the means and volatilities of the investments available in the non-annuitized portion of the portfolio, and deferring the start of annuity payments.

4. In contrast however, we find that investors who are able to annuitize all of their consumption needs are best off doing so, and if they are unable to completely annuitize their consumption needs, then they are best off annuitizing as much as possible, since the more they annuitize, the higher their probability solvency becomes, at least with annuities whose payouts start immediately. That said, if the investor decides that losing some probability solvency is worthwhile to create a higher expected bequest, we provide a number of tables that allow them to do this in the way that best fits their needs. Specifically, these tables help the investor select annuity characteristics like the potential deferral of starting payouts, potential inflation adjustments to payouts once they start, and potentially having a bequest clause to give some protection to heirs if the investor passes away early, so the investor's chosen annuity best suits their desire to balance longevity and legacy risk given their chosen lifestyle. For investors who are able to obtain a 100% solvency probability, these tables enable locating the annuity that maximizes their expected bequest, meaning their legacy risk is minimized subject to having no longevity risk.

We note that the definition of "consumption needs" in this paper is investor dependent. That is, an investor might define consumption needs as what they would ideally like to spend each year, or what they project they will likely spend each year, or what they believe they must spend each year to maintain a minimally acceptable lifestyle. Our results must therefore be interpreted in light of which definition the investor wishes to consider.

The paper proceeds as follows: Section 2 briefly summarizes some previous literature on annuities in light of our approach. Section 3 presents the framework for our approach and analysis. Section 4 presents several illustrative results and their implications. We conclude in Section 5.

2 Previous Literature

In this section, we look at some of the previous approaches to annuities in the context of the approach we take in this paper. There is a considerable body of research on annuities and how best to use them. While no source could cover all the work on the subject, Richards et al. (2004) and Milevsky (2013) offer excellent reviews of the nature of longevity risk and the arguments in favor of various types of annuities.

Annuities are optimal for an investor with fixed living costs and no bequest motive (Yaari, 1965), although Horneff et al. (2007) offers a dissenting view that advocates for variable annuities and annuitization in a gradual manner. Annuities have disadvantages. They generally reduce bequests substantially (that is, they create more legacy risk), and they also pose liquidity risk as the cash inflows from annuities do not always match the uncertain cash outflows that arise over the course of life. The more annuities an investor purchases, the more they fail to realize gains from good years in the market that can build a financial buffer for later years. Further, provider risk must be taken into account, especially over a very long horizon.

On the other hand, a pure investment portfolio approach to retirement with no fail-safe income sources like annuities may incur greater longevity risk due to its volatility. This can especially be the case if the investor relies on simple rules of thumb, such as the 4% rule in which the investor consumes 4% of the worth of their portfolio in the first year and increases this amount by inflation in each successive year (Bengen, 2004; Cooley et al., 1998; Finke et al., 2013), or if the investor relies on target-date funds, whose glide paths may not be optimal (Guillemette, 2017). In this paper, we seek to minimize the inherent risks in an investment portfolio by using dynamic programming to optimally change investments each year so that this non-annuitized portion of the investor's portfolio maximizes the chance of the investor staying solvent throughout their lifetime. Further, this optimization takes into account the effect of an annuity if there is one.

Our dynamic programming strategy dynamically hedges longevity risk with evolution models that best fit investment portfolios containing instruments like stocks and bonds. Static hedging of longevity risk, on the other hand, has been considered with many instruments, including (1) survivor bonds (Blake and Burrows, 2001), (2) longevity bonds, swaps, and derivatives, where the payouts are indexed to mortality (Ngai and Sherris, 2011), (3) forward-start inflation indexed securities, see SELFIES (Merton and Muralidhar, 2017), and (4) retirement bonds, such as those designed in Martellini et al. (2019) and Martellini et al. (2020). These approaches are generally evaluated via forward simulations, whereas dynamic programming is approached via backward recursion, which is deterministic and computationally efficient, in addition to generating optimal dynamic, instead of static, strategies.

A comparison of optimal decumulation (that is, withdrawal rate) strategies when there is no bequest motive is undertaken in Blanchett et al. (2012) using forward simulation, assuming a power law (i.e., constant relative risk aversion) utility function to measure the value of the annual withdrawals. In contrast, we favor a preset consumption stream, where risk is measured by insolvency probability. That is, because we take a goals-based perspective, we are not interested in optimizing the utility of the money taken from the portfolio. We are instead interested in optimizing the probability that the investor remains solvent while achieving their consumption stream goal, which does not rely on specifying a utility function.

Regarding more common decumulation strategies, a subcase of our formulation can conform to the 4% rule (Bengen, 2004) with a projected rate of inflation, although it improves on the 4% rule in key ways, including optimally dynamically changing investments to minimize longevity risk, as well as optimally incorporating annuities, which the 4% rule does not address. Another common decumulation strategy is simply to follow the IRS-imposed required minimum distributions (RMDs), which Blanchett et al. (2012) and Blanchett (2013) find works better than the 4% rule. The RMD strategy has no longevity risk, but it has considerable lifestyle risk, specifying considerable changes to an investor's consumption stream over the long term and also, when there are market changes, in the short term. The RMD method is not a goals oriented strategy, nor, as with the 4% rule, is it designed to work with, or say anything about, annuities.

We restrict ourselves in this paper to fixed annuities (that is, annuities with deterministic payouts while the investor is alive). A more detailed analysis would also allow for the possibility of variable annuities[2] (where stochastic payout amounts depend on an underlying market). These variable annuities can also come with a guaranteed lifetime withdrawal benefit (GLWB) implemented

[2] https://www.investopedia.com/terms/v/variableannuity.asp

through a guaranteed minimum withdrawal benefit (GMWB) clause.[3] The GMWB rider enables retirees to take more risk in their portfolio accounts to retain some market upside, though there will be correlation between the annuity portfolio and the investor's own investment portfolio, so that this upside may not be available when it is most needed.

In this paper, the decision to purchase a fixed annuity is made strictly at the current time. The investor may choose a single-premium immediate annuity (SPIA), where the payouts begin immediately, or they may choose a deferred immediate annuity (DIA)[4] (such as a QLAC), where the initial payout is deferred for a specified number of years. The value of a DIA depends on the long end of the yield curve, so a full analysis of a DIA would need to consider the yield curve's stochasticity, as undertaken in Huang et al. (2017) in a continuous-time utility framework.

Our requirement that the annuity decision be made at the current time ignores the fact that deferring the decision to annuitize also has value, considered in Milevsky (1998) and Milevsky and Young (2001). We leave adaptation of our dynamic programming approach to consider this real option and understand how much value it has to later work. This real option is mostly ignored in the literature but is important to consider as its value depends on medical advancements countervailed by increasing risks in the labor market around human capital, and requires a much more detailed treatment in a follow-up paper.

In this paper, we explore an algorithm for determining the best mix of annuities and a dynamically rebalanced portfolio so as to achieve a balance between managing longevity risk while maintaining a healthy standard of living, and giving due consideration to the bequest motive (see arguments for a smaller SPIA in the mix by Kitces and Pfau (2013)). By assessing the best mix at different consumption levels, we enable the investor to make knowledgeable decisions about the size and other characteristics of the annuity they decide to purchase, including possible deferral of the initial annuity payout, possible inflation adjustments to the annuity's payouts once they start, and possible inclusion of bequest clauses to protect heirs if the investor passes away early. These decisions are critical because the annuity locks in the investor once and for all; it is a decision that is not reversible, nor one that is easily unwound through portfolio rebalancing. Given that the aging investor faces market risk, longevity risk, and inflation risk, retirement planning with annuities is a long-horizon problem, calling for careful analysis. For a good overview of several issues related to longevity risk and financial planning, see Finke and Blanchett (2016).

Our paper complements existing work on annuities by considering in more detail the trade-offs between investor solvency and bequest size, giving greater attention to legacy motives than in the extant literature. Further, it shows how we can take advantage of recent techniques for rebalancing investment portfolios so that the non-annuitized portion of the portfolio can optimally complement annuity purchases, instead of working separately from them.

3 Details Of The Model

In our model, two decisions are initially made:

[3] https://www.investopedia.com/terms/g/glwb.asp

[4] DIAs are becoming more popular since a recent decision by the US Treasury department (July 2014) to permit the purchase of DIAs inside tax-sheltered retirement plans and to exempt them from certain distributional requirements at age 72, as long as the annuity commences payouts prior to age 85, see Huang et al. (2017).

3.1 The Consumption Stream

The first decision is choosing an annual consumption stream for the investor over their lifetime, which we will also refer to as a "standard of living," a "lifestyle," or, especially in Section 4's tables, "living expenses." While the consumption stream can be any desired function of time, our examples in Section 4 will generally let this be a given initial amount that is then adjusted by a 2% inflation rate in future years, i.e., a fixed amount in real dollars. We note that the 4% rule (Bengen, 2004) is an example of this model of a fixed amount in real dollars.

3.2 The Annuitized Portion Of The Portfolio

The second decision that is initially made is choosing which annuity the investor wants to purchase. There are a plethora of annuities that can be selected due to a wide range of available features. The typical questions that must be answered to determine these features are: (1) Is the annuity payout adjusted for inflation? If so, is the rate of increase fixed in advance? Or is it indexed to CPI? Is it capped? (2) Is the annuity a SPIA or a DIA? (3) Is there a bequest clause? If so, what is its nature? (4) Is there a GLWB clause? (5) Is the annuity fixed term or mortality based? (6) Do we have a variable annuity or a fixed annuity?

In this paper we will consider annuities with the following features: (1) We assume a fixed rate of inflation, which we may or may not use. When it is used, we will set it to be 2%, so that it matches the inflation rate assumed for the consumption stream. We do not examine more complicated models for inflation in this paper, although we could easily expand our model to, say, a constant inflation rate that differs from the consumption stream's rate of inflation. (2) We will consider both SPIA and a variety of DIA models. (3) We will consider the effect of selecting one of the most common bequest clauses, which pays the heirs a lump sum equal to any positive difference between the nominal price paid to purchase the annuity and the nominal payouts made prior to the annuity holder passing away. (4) While our model can be very easily altered to accommodate a GLWB clause, we will not explore it here. (5) Similarly, while our model can also be very easily altered to accommodate annuities with a fixed term, we will not explore it here. (6) We will only consider fixed annuities; variable annuities require a more complex analysis, which we leave to subsequent papers.

We will assume that we know the cost of each annuity and that we know how much each annuity pays out each year until the investor passes away. The date of the investor passing, however, is unknown, so we must use mortality tables to determine the likelihood of the investor's passing each year.[5]

3.3 The Non-Annuitized Portion Of The Portfolio

Once a consumption stream and an annuity have been selected, we are able to compute the remaining consumption needs, as well as the initial size of the non-annuitized portion of the portfolio that must address these remaining consumption needs. We note that if the annuity payment is bigger than the consumption, then the difference will become an infusion into the non-annuitized portion of the portfolio. We are easily able to accommodate any additional projected external infusions over time into the non-annuitized portion of the portfolio, such as

[5]Alternatively, see Huang et al. (2017) for a fascinating analysis of thinking about mortality as a function of stochastic biological time.

Social Security income, although, in the interests of simplicity, this paper's examples do not explore this. (This can be accommodated by simply altering $C(t)$ in Appendix A to reflect these external infusions.)

With the remaining consumption needs and infusions known, we can use the backwards recursion method of dynamic programming (Bellman, 1952) to determine the optimal strategy for the non-annuitized portion of the portfolio. Specifically, we determine at each time and wealth value, the optimal investment portfolio that maximizes the investor's solvency probability (i.e., minimizes longevity risk) in light of the fact that we can change the investment portfolio in each future year.

The details of the solution to this dynamic programming algorithm are given in Appendix A, which follows the approaches used in Das et al. (2020) and Das et al. (2021). As a quick summary, the optimal dynamic portfolio problem is solved using backward recursion on a state space comprising wealth and time. The optimal investment portfolio strategy at each point in the state space is determined from a set of candidate investment portfolios that generally lie on the efficient frontier, but are not restricted to this. We note that the optimization takes into account the effect of payouts from the annuity, possible infusions, planned annual expenses, and mortality risk.

The fact that our optimization takes all of these factors into account and is optimizing solvency can have a huge positive impact on the investor's outcome. In contrast, target date funds only take the age of the investor into account. Section 4.3.5 of Das et al. (2020) compares our dynamic programming approach to a common target date fund glide path for a 50 year old investor with \$100,000 who contributes \$15,000 real dollars to their nest egg each year until they retire at age 65, after which they consume \$50,000 real dollars in each of the next 15 years. If the investor chooses the target date fund, they have a 26.6% chance of remaining solvent, but if they optimize their investment portfolio choice by using our dynamic programming strategy, the chance of remaining solvent becomes 58.6%, a considerable improvement.

Once we know the optimal portfolio investment strategy, it is straightforward to determine the probability distribution for each of the wealth states (including bankruptcy) occurring in any given year. Combining this knowledge with the mortality tables allows us to compute the expected present value of the bequest given a consumption stream rate and a specific annuity.

3.4 Determining The Best Annuity and Consumption Stream

Given that both the annuitized and the non-annuitized portions of the portfolio are designed to keep the investor solvent, it is unclear what allocation between these two portions will best attain the investor's solvency goal. For typical market costs for annuities, is it optimal to annuitize all, none, or a specific fraction of the investor's portfolio? This is one of the key questions we look to answer in this paper.

If the answer is that it is best to annuitize some or all the investor's portfolio to maximize solvency probability, the investor may still, as a secondary consideration, want to know how much the annuity decreases the expected present value of their bequest. For example, an investor may find that by purchasing a large annuity, they have a 99.8% chance of remaining solvent for their lifetime, but if they choose an annuity that is only 80% as large, they will still have a 99.7% chance of remaining solvent, but their expected bequest will increase by \$40,000. Which size annuity they should select is a matter of personal preference, but it should be unarguable that the investor should be able to see these trade-offs so they are able to make an informed choice.

Therefore, we will present both the solvency probabilities and the expected present value of the bequest for a variety of annuity choices and consumption stream rates. This allows the investor to knowledgeably choose the consumption stream rate and annuity that best fits their desired outcomes.[6] Because these choices are inherently individual, they are choosing from a Pareto frontier that we call the "solvency-bequest frontier." This choice enables an investor to balance the trade-offs between lifestyle risk, longevity risk, and legacy risk in a manner that is optimized where it can be and personalized where it must be.

The next section details examples of these solvency-bequest frontiers and their implications.

4 Examples And Analysis Of Results

4.1 External inputs

We detail our market assumptions for the cost of annuities, for the investment portfolios available in the non-annuitized portion of the portfolio, and for mortality:

Annuity costs: We consider typical rates for purchasing annuities in Table 1. For each of Table 1's three panels, the investor is assumed to be 65 years old and can choose an annuity where the payouts start immediately (an SPIA) or where the first payout is deferred until the investor is 70, 75, 80, or 85 years old (a DIA). Payouts, once started, continue annually until the investor passes away. The top panel is for annuities whose initial payments are increased each year by a 2% inflation rate. So, for example, if the investor pays $250,000 for a deferred annuity that starts at age 70, they will receive the value in the table, $16,891, the year they are 70, then receive $16,891 $\times$ 1.02 when they are 71, $16,891 $\times (1.02)^2$ when they are 72, etc. The middle panel is for annuities whose initial payments stay constant, so the value in the table is the payment received every year once the annuity payments start. The bottom panel is for annuities that, as in the top panel, have initial payments that increase each year by a 2% inflation rate. Additionally, these annuities contain a bequest clause that returns to the heirs any positive difference between the upfront nominal lump sum paid to purchase the annuity and the nominal payouts made prior to the investor's passing away.

Note, as expected, that the payout amounts are proportional to the cost of the annuity. That is, within each of the three panels, the payout from the annuity costing $500,000 is twice the payout from the annuity costing $250,000; the payout from the annuity costing $750,000 is thrice the payout from the annuity costing $250,000; and so on.

[6] We note that while the annuity has the further disadvantage of inflexibility in regards to sudden spending needs, we will not consider this further due to the difficulty in quantifying this disadvantage. Similarly, we do not consider the fact that as the investor gets closer to bankruptcy, they might look to decrease their consumption, since the amount and negative effect of this potential decrease is also inherently difficult to quantify.

Inflation adjustment = 2% | Bequest refund = No					
	Annuity Start Age				
Cost	65	70	75	80	85
$2,000,000	$93,230	$135,130	$210,370	$352,520	$664,160
$1,750,000	$81,576	$118,239	$184,074	$308,455	$581,140
$1,500,000	$69,923	$101,348	$157,778	$264,390	$498,120
$1,250,000	$58,269	$84,456	$131,481	$220,325	$415,100
$1,000,000	$46,615	$67,565	$105,185	$176,260	$332,080
$750,000	$34,961	$50,674	$78,889	$132,195	$249,060
$500,000	$23,308	$33,783	$52,593	$88,130	$166,040
$250,000	$11,654	$16,891	$26,296	$44,065	$83,020
$0	$0	$0	$0	$0	$0

Inflation adjustment = none | Bequest refund = No					
	Annuity Start Age				
Cost	65	70	75	80	85
$2,000,000	$115,820	$163,520	$247,290	$402,860	$740,360
$1,750,000	$101,343	$143,080	$216,379	$352,503	$647,815
$1,500,000	$86,865	$122,640	$185,468	$302,145	$555,270
$1,250,000	$72,388	$102,200	$154,556	$251,788	$462,725
$1,000,000	$57,910	$81,760	$123,645	$201,430	$370,180
$750,000	$43,433	$61,320	$92,734	$151,073	$277,635
$500,000	$28,955	$40,880	$61,823	$100,715	$185,090
$250,000	$14,478	$20,440	$30,911	$50,358	$92,545
$0	$0	$0	$0	$0	$0

Inflation adjustment = 2% | Bequest refund = Yes					
	Annuity Start Age				
Cost	65	70	75	80	85
$2,000,000	$81,540	$119,930	$173,920	$267,790	$465,610
$1,750,000	$71,348	$104,939	$152,180	$234,316	$407,409
$1,500,000	$61,155	$89,948	$130,440	$200,843	$349,208
$1,250,000	$50,963	$74,956	$108,700	$167,369	$291,006
$1,000,000	$40,770	$59,965	$86,960	$133,895	$232,805
$750,000	$30,578	$44,974	$65,220	$100,421	$174,604
$500,000	$20,385	$29,983	$43,480	$66,948	$116,403
$250,000	$10,193	$14,991	$21,740	$33,474	$58,201
$0	$0	$0	$0	$0	$0

Table 1: Initial annual annuity payouts for an investor who is currently 65 years old and purchases an annuity now, given a variety of possible times for the payouts to start and a variety of levels for the cost of the annuity purchase. (i) The annuities in the top panel have initial payouts that grow by 2% each year. (ii) The annuities in the middle panel have payouts that do not change over time, so they have a larger initial payout than in the top panel. (iii) The bottom panel has the lowest initial payouts, since the initial payouts not only grow at 2%, as in the top panel, but there is also a bequest refund clause; specifically, the annuity refunds to the heirs anything left from the nominal cost of the annuity after deducting the nominal value of all payouts made prior to the investor's passing.

Investment Portfolio Options In The Non-Annuitized Portion Of The Portofolio: We will assume for our examples that each possible investment portfolio evolves by geometric Brownian motion (although any Markovian distribution is just as easily accommodated). Given this, we only need to specify the mean (μ) and volatility (σ) of each investment portfolio's returns. Preferably these (μ, σ) pairs are on the efficient frontier, although this is not required. We will assume the investor has access to each of the 15 investment portfolio options contained in Table 2. The dynamic programming strategy discussed in Subsection 3.3 allows us to optimally switch between these 15 investment portfolios every year so as to optimize the investor's probability of remaining solvent. Note that these 15 investment portfolios are quite conservative: the most aggressive has a μ of only 5.28%. This can make the annuity more attractive by comparison. We will later look at the effect of increasing the μ values or the σ values by a given amount in all 15 investment portfolios. This will provide some understanding for how robust our results are to errors in forecasting μ and σ.

Portfolio	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
μ (%)	2.64	2.83	3.02	3.21	3.40	3.59	3.77	3.96	4.15	4.34	4.53	4.72	4.90	5.09	5.28
σ (%)	3.70	4.11	4.63	5.22	5.86	6.53	7.23	7.95	8.68	9.43	10.18	10.94	11.71	12.47	13.25

Table 2: The expected returns (μ) and volatilities (σ) for 15 possible investment portfolios.

Mortality Tables: Recall from Section 3 that we need mortality tables both to optimize the investment portfolio strategy in the non-annuitized portion of the portfolio and to determine the expected present value of the bequest. In our examples, we have used the unisex mortality tables published by the Internal Revenue Service, which are shown in Figure 1. These give the conditional probability of mortality, meaning the probability of dying in the next year conditioned on being alive at the beginning of the year. Due to the rarity of individuals living past the age of 115, these tables assume the conditional probability of mortality is 0.5 between the ages of 115 and 120. Because living past 120 is even more rare, the table assumes no one lives past the age of 120 — despite occasional exceptional cases like Jeanne Calment from the Introduction!

4.2 Investor-specific inputs

We assume the following investor-specific information for the remaining inputs that are needed to run our examples:

Age: We assume the investor is currently 65 years old to fit with the annuity costs presented in the previous subsection.

Initial wealth: We assume the investor currently has $2,000,000. They can choose to initially allocate all, some, or none of this to an annuity.

Consumption Rates: The investor will choose from a set of eight discrete consumption rates that range from $50,000 to $120,000 in steps of $10,000. These consumption rates are the initial annual amounts. They will increase by 2% each year regardless of the annuity chosen by the investor, although, obviously, if the investor chooses an annuity that increases its payouts by 2% each year, that will match the consumption increase rate.

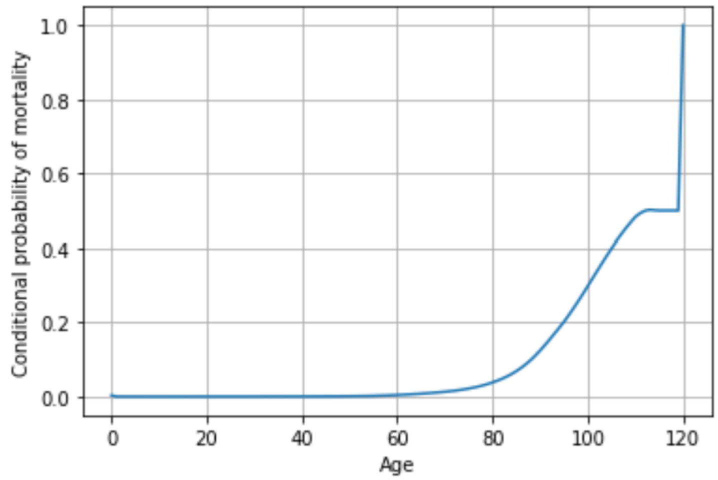


Figure 1: Mortality as a function of age. The mortality probabilities in the graph are conditional, i.e., they give the probability of passing away in a given year conditioned on being alive at the beginning of the year. The values shown are based on a unisex table from the Internal Revenue Service (IRS); see https://www.irs.gov/pub/irs-drop/n-19-26.pdf. The table assumes that the maximum age is 120 years.

Annuity Choice: The investor will only consider the annuities presented in Table 1. Note that these are all priced for a 65 year old, so if the investor purchases an annuity, it will be now, not later, even if the annuity they purchase now may be a DIA whose payouts do not start until later.

4.3 Illustrative Examples

4.3.1 Our Base Case Annuities: Payments Start Immediately, Payouts Grow At 2% Each Year, No Bequest Clause

Our base case annuities correspond to the first column of the top panel in Table 1. That is, a 65 year old retiree selects an annuity to purchase at one of the variety of annuity prices in the table and begins to receive payouts from this annuity immediately (an SPIA). The initial payout amounts shown in Table 1 grow each year by 2% and continue until the retiree passes away. When the retiree passes away, the annuity pays nothing to the heirs. In later subsections, we will see the effect on these base case annuities if (1) we keep the payouts constant instead of increasing them by 2% each year, or (2) we add a bequest clause that pays the heirs any money left over from the nominal initial cost of the annuity after removing the nominal cost of the payouts made before the retiree passed away, or (3) for each of the 15 possible investment portfolios shown in Table 2, the expected return is increased by 2 percentage points or the volatility is increased by 4 percentage points, representing errors in projecting μ or σ, or (4) we defer payouts from the annuity until the retiree is 70, 75, 80, or 85.

For each of our base case annuities, we see in Figure 3 both the solvency and the expected present value of the bequest given various consumption rates. The upper panel in the figure, which generally will not qualitatively change much in later subsections, allows us to answer the key question we posed earlier: Should we annuitize all, none, or a specific fraction of the investor's portfolio if we want to optimize the probability of lifetime solvency for the investor?

Solvency Probability

Cost of the Annuity	$50,000	$60,000	$70,000	$80,000	$90,000	$100,000	$110,000	$120,000
$2,000,000	100.0%	100.0%	100.0%	100.0%	100.0%	0.0%	0.0%	0.0%
$1,750,000	100.0%	100.0%	100.0%	100.0%	94.5%	36.5%	14.1%	8.0%
$1,500,000	100.0%	100.0%	100.0%	100.0%	85.7%	53.3%	30.5%	18.5%
$1,250,000	100.0%	100.0%	100.0%	98.4%	82.2%	60.2%	41.7%	28.7%
$1,000,000	100.0%	100.0%	99.9%	94.9%	80.4%	63.8%	48.8%	36.8%
$750,000	100.0%	100.0%	98.8%	91.7%	79.4%	66.0%	53.5%	42.8%
$500,000	100.0%	99.8%	97.0%	89.2%	78.6%	67.5%	56.9%	47.3%
$250,000	100.0%	99.0%	94.9%	87.4%	78.1%	68.6%	59.3%	50.8%
$0	99.7%	97.8%	93.0%	85.9%	77.7%	69.4%	61.2%	53.6%

Initial annual living expenses (which then increase by inflation)

Solvency Probability (%): 0.0 – 1.0

Expected Bequest (Present Valued)

Cost of the Annuity	$50,000	$60,000	$70,000	$80,000	$90,000	$100,000	$110,000	$120,000
$2,000,000	$953,000	$751,000	$529,000	$301,000	$74,000	$0	$0	$0
$1,750,000	$1,005,000	$777,000	$550,000	$322,000	$123,000	$50,000	$17,000	$9,000
$1,500,000	$1,026,000	$799,000	$571,000	$344,000	$225,000	$149,000	$82,000	$47,000
$1,250,000	$1,047,000	$820,000	$592,000	$404,000	$327,000	$252,000	$173,000	$115,000
$1,000,000	$1,069,000	$841,000	$619,000	$494,000	$429,000	$355,000	$272,000	$202,000
$750,000	$1,090,000	$863,000	$685,000	$595,000	$531,000	$457,000	$374,000	$297,000
$500,000	$1,111,000	$899,000	$771,000	$695,000	$634,000	$560,000	$476,000	$396,000
$250,000	$1,135,000	$969,000	$865,000	$799,000	$736,000	$663,000	$579,000	$497,000
$0	$1,180,000	$1,052,000	$965,000	$903,000	$837,000	$765,000	$682,000	$599,000

Initial annual living expenses (which then increase by inflation)

Expected Present Value of Bequest, in $ (1e6): 0.0 – 1.0

Table 3: Solvency-bequest trade-offs for our base case annuities. In this example, the investor is 65 years old, has a current wealth of $2 million, and buys an annuity whose payouts being immediately and then increase by 2% each year. The figure shows the solvency probability in the top panel and the expected present value of the bequest in the bottom panel for a range of annuity prices and for a range of living expenses (i.e., consumption rates) that, like the payouts, increase by 2% each year. The annuities presented here do not include a bequest refund when the investor passes away.

The top panel in Table 3 shows that if the retiree has initial living expenses that are $90,000 or lower, the investor should annuitize as much of their living expenses as possible. However, if the retiree has initial living expenses that are $100,000 or higher, the investor should annuitize none of their portfolio. In particular, it is optimal to annuitize all or none of the portfolio, never some of it.

Why does the behavior change between $90,000 and $100,000? Since both the annuity payouts and the consumption rates are assumed to increase at exactly 2%, we actually know the exact number at which it changes: $93,230, the annual payout generated at age 65 if the entire portfolio is annuitized, as shown in the upper left-hand corner of the table in the top panel of Table 1. Any consumption rate below this number is completely addressed by full annuitization, while any rate above this cannot be completely addressed by full annuitization and causes the retiree to become insolvent in their first year, which is reflected by the top row of the Solvency Probability panel in Table 3, where the solvency probability is either 100% or 0%.

What is more interesting is that this pattern holds when the cost of the annuity is reduced. That is, if the retiree is *capable* of annuitizing all of their expenses, then the bigger the annuity, the better, but if they are not, then the smaller the annuity, the better.

This observation has important implications for advising retirees considering annuities. Franco Modigliani, in his Nobel Prize acceptance lecture,[7] discussed the apparent low levels of annuitization in the US. He wrote—

> "Indeed, in view of the practical impossibility of having negative net worth, people tend to die with some wealth, unless they can manage to put all their retirement reserves into life annuities. However, it is a well known fact that annuity contracts, other than in the form of group insurance through pension systems, are extremely rare. Why this should be so is a subject of considerable current interest and debate (see, e.g., Friedman and Warshawsky (1985a); Friedman and Warshawsky (1985b))."

This rarity of individual annuity contracts, which continues to today,[8] is known as the "annuitization puzzle." Experts in behavioral economics and finance have looked to explain this phenomenon. See, for example, Hu and Scott (2007) and Benartzi et al. (2011).

Our results, however, show that there is a rational explanation in addition to the behavioral explanations for eschewing annuities for the large class of investors with considerable consumption needs. For these investors, annuities not only reduce their bequest (as expected but now quantified in the lower panel of Table 1), but also their probability of remaining solvent. Annuitization is therefore best considered by the well-off investor, not the poor investor, at least if the investor's goal is to optimize their probability of remaining solvent given fixed consumption rates.

While increased annuitization increases the solvency probability for well-off investors, they must balance this with the fact that increased annuitization also decreases the expected present value of their bequest. This balancing is made possible by using both panels in Table 3.

For example, an investor with initial living expenses of $70,000 is guaranteed to stay solvent if they annuitize all their spending, which would cost $1,501,652. (This cost is determined by interpolating the results for the annuities that cost $1,500,000 and $1,750,000 in the first column of the top panel in Table 1.) But by looking at the $70,000 columns in both panels of Table 3, we realize that if the investor instead purchased an annuity that only cost $1,250,000, they would still have a 100.0% solvency probability, but would gain a little more than $21,000 in the expected present value of their bequest. Or, if they reduced the annuity to one that only cost $1,000,000 instead of $1,501,652, they would still maintain a 99.9% solvency probability, but would gain a little more than $48,000 in the expected present value of their bequest compared to annuitizing all their spending. For many investors, even those whose original stated goals were to maximize solvency probability, the trade-off to obtain the additional expected bequest money might well be judged to be worthwhile.

[7] https://www.nobelprize.org/uploads/2018/06/modigliani-lecture.pdf

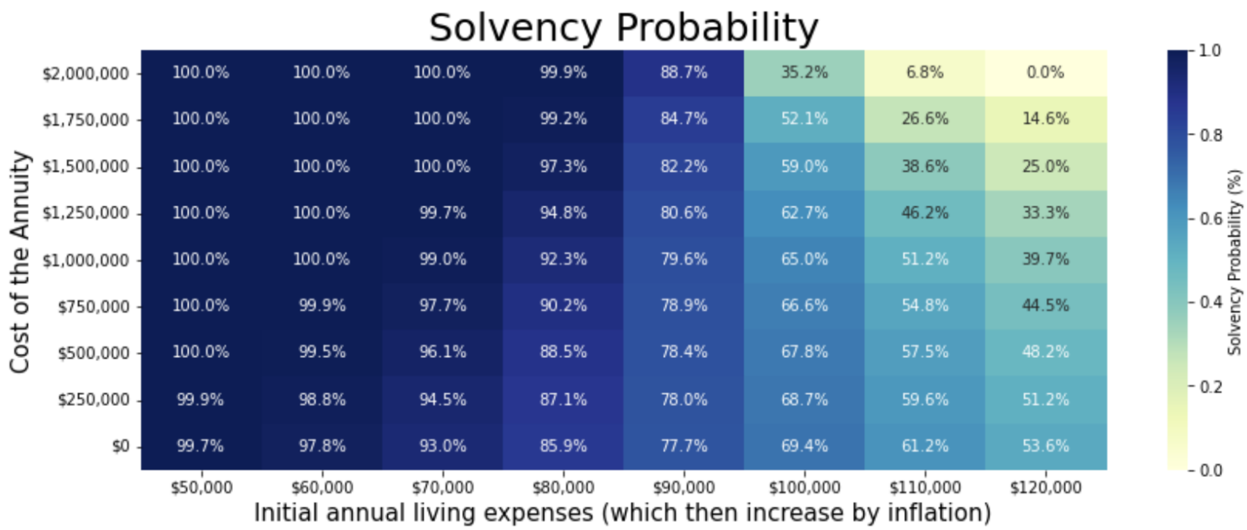
[8] https://www.marketwatch.com/story/why-dont-retirees-like-annuities-11652296853. An interesting aspect of retirement plans is that they focus on accumulation and little attention is paid to the decumulation phase and its risks, such as insolvency, for which annuities are a possible solution. Decumulation is barely optimized, left to thumb-rules such as the 4% rule or the IRS mandated required minimum distributions.

These choices for how to best balance solvency probability with the expected present value of a bequest innately depend on individual preferences. In this regard we have a Pareto frontier, which we refer to as the "solvency-bequest frontier." That is, the two panels in Table 3 form this "solvency-bequest frontier," which enables investors to knowledgeably balance their desire for solvency probability with their desire (if any) to have a larger bequest when they pass away.

In the remainder of this subsection, we consider the effect of changing individual characteristics of our base case annuities.

4.3.2 Constant Payout Annuities

What happens if the yearly annuity payouts are held constant over time instead of increasing annually at a rate of 2%? The constant annuity payout amounts are given in the first column of the middle panel of Table 1, which, of course, have higher values than the corresponding initial payouts that increase annually by 2% contained in the first column of the top panel of Table 1. The solvency-bequest frontier for these constant payout annuities is given in Table 4.



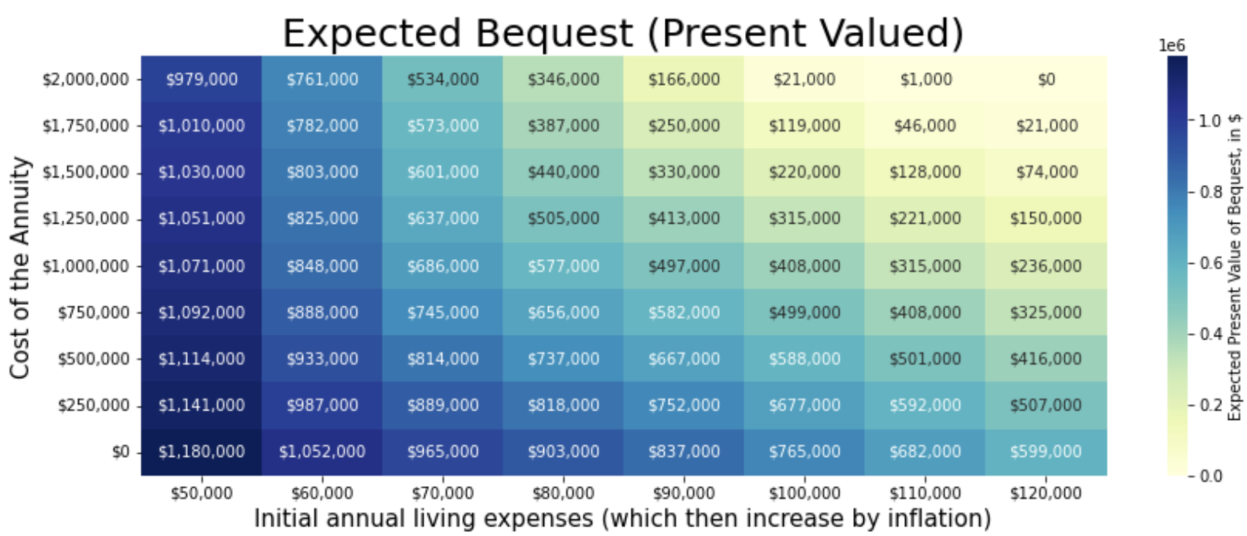


Table 4: The solvency-bequest frontier when the annual annuity payouts remain constant. The initial annual living expenses are still assumed to increase by 2% each year.

We compare and contrast the results in Table 4 to the results for the base cases in Table 3. First we note that, as in our base cases, we maintain our key observation that the solvency probability increases with more annuitization if the consumption rate is $90,000 or below, but it decreases with more annuitization if the consumption rate is $100,000 or above. So, as before, we conclude that retirees with high consumption rates (or more specifically an inability to annuitize their consumption) should eschew annuities.

While these retirees with high consumption rates are best off completely eschewing annuities, we do note that they are better off with the constant payout annuity than with the base case 2% annual payout increase annuity, because the solvency probabilities are higher and, at the same time, the expected bequests are higher. On the other hand, for retirees with consumption rates that are $90,000 or below, switching to the constant payout annuity leads to lowering the solvency probability but increasing the expected bequest. So, again, the preference between the base cases and constant payout annuities becomes a matter of personal preference, and these tables enable the investor to make these trade-offs knowledgeably.

Because the payouts are constant, we note that the top row, for which the entire portfolio is used to purchase the annuity, is no longer either 100% or 0%, as it was for the base cases. For this row, the constant payout rate is $115,820, as shown in the upper left-hand entry in the middle panel of Table 1. If the consumption rate is higher than that, then the retiree becomes bankrupt immediately, as we can see in the upper right-hand corner of the top panel in Table 4. But consider the case where the entire portfolio is used to purchase the annuity and the consumption is, say, $100,000. Initially, the $115,820 payout pays for all the consumption, and then the remaining $15,820 becomes the non-annuitized portion of the portfolio. Since we assume consumption is increasing at a 2% rate, eventually the $115,820 is not large enough to satisfy all of the retirees consumption needs. At this point, the retiree must pay for any remaining consumption needs by using the non-annuitized portion of their portfolio due to optimally investing the excesses from the previous annuity payments. Eventually, of course, these funds become exhausted by the ever-increasing consumption needs. Should the investor pass away before this happens, they have succeeded in remaining solvent for their lifetime. Table 4 shows that there is a 35.2% chance that this will occur.

The fact that so much more money comes to the investor in the early years increases the expected bequest. This increase more than makes up for the generally slight reduction in solvency probability (when a reduction occurs), which decreases the expected bequest. This results in the observed overall increase to all the expected bequest values seen in the bottom panel of Table 4 when compared to the bottom panel for the base cases in Table 3.

4.3.3 Inclusion Of A Bequest Clause

What happens if we add a bequest clause to our base cases? The annuity payouts now correspond to the first column of the bottom panel of Table 1, which, to account for the bequest clause, have lower values than those for the base cases in the first column of the top panel of Table 1. Recall that when the retiree passes away, the bequest clause provides to the heirs any money left over from the initial cost of the annuity minus all annuity payouts during the retiree's lifetime, with no adjustments for inflation to any of these numbers.

The resulting solvency-bequest frontier is given below in Table 5. The expected bequest in the lower panel of Table 5 is computed, as before, by providing no money to the heir if the retiree becomes bankrupt during their lifetime, but now we make an exception for any money from the bequest clause, which is included in the expected bequest whether or not the investor becomes bankrupt.

Solvency Probability

Cost of the Annuity	$50,000	$60,000	$70,000	$80,000	$90,000	$100,000	$110,000	$120,000
$2,000,000	100.0%	100.0%	100.0%	100.0%	0.0%	0.0%	0.0%	0.0%
$1,750,000	100.0%	100.0%	100.0%	93.5%	35.5%	13.8%	7.9%	5.5%
$1,500,000	100.0%	100.0%	100.0%	89.4%	57.0%	32.6%	19.6%	13.1%
$1,250,000	100.0%	100.0%	99.6%	87.9%	66.0%	46.1%	31.7%	22.4%
$1,000,000	100.0%	100.0%	98.3%	87.1%	70.6%	54.7%	41.4%	31.2%
$750,000	100.0%	99.9%	96.6%	86.6%	73.5%	60.4%	48.6%	38.8%
$500,000	100.0%	99.4%	95.2%	86.3%	75.4%	64.3%	53.9%	44.8%
$250,000	99.9%	98.7%	94.0%	86.1%	76.7%	67.2%	58.0%	49.7%
$0	99.7%	97.8%	93.0%	85.9%	77.7%	69.4%	61.2%	53.6%

Initial annual living expenses (which then increase by inflation)

Expected Bequest (Present Valued)

Cost of the Annuity	$50,000	$60,000	$70,000	$80,000	$90,000	$100,000	$110,000	$120,000
$2,000,000	$996,000	$772,000	$544,000	$317,000	$282,000	$282,000	$282,000	$282,000
$1,750,000	$1,019,000	$791,000	$564,000	$368,000	$295,000	$263,000	$256,000	$253,000
$1,500,000	$1,038,000	$810,000	$583,000	$443,000	$371,000	$300,000	$261,000	$243,000
$1,250,000	$1,057,000	$830,000	$614,000	$519,000	$450,000	$369,000	$305,000	$263,000
$1,000,000	$1,076,000	$849,000	$676,000	$596,000	$530,000	$447,000	$370,000	$310,000
$750,000	$1,096,000	$875,000	$745,000	$672,000	$607,000	$526,000	$444,000	$373,000
$500,000	$1,115,000	$927,000	$816,000	$748,000	$682,000	$606,000	$522,000	$444,000
$250,000	$1,140,000	$985,000	$891,000	$827,000	$760,000	$686,000	$602,000	$520,000
$0	$1,180,000	$1,052,000	$965,000	$903,000	$837,000	$765,000	$682,000	$599,000

Initial annual living expenses (which then increase by inflation)

Table 5: The solvency-bequest frontier when a bequest clause is added to the annuity. The bequest clause refunds to the heirs any money left over from the initial purchase of the annuity after subtracting all payouts made from the annuity while the retiree was living.

As expected, the reduced annual payouts due to the bequest clause cause the solvency probabilities in the upper panel of Table 5 to be smaller than those in the base cases shown in the upper panel of Table 3. Further, since the initial annual payout for the retiree annuitizing their entire $2,000,000 savings is reduced from $93,230 in the base case to $81,540 in the case of adding the bequest clause (which comes from the upper left-hand entries in Table 1's top and bottom panels), we now have that the retiree becomes immediately bankrupt if their living expenses are initially $90,000. Note that this change is reflected in the $90,000 column in Table 5, which confirms our previous observation that even partial annuitization is unwise for investors seeking to maximize their solvency probability if they are unable to completely annuitize all of their living expenses.

Unsurprisingly, the bequest clause causes the expected bequest to increase. The specific magnitudes for these increases can be seen by comparing the bottom panel of Table 5 with the bottom panel for the base cases in Table 3. Note that the largest increases, which can be as high as $282,000, occur in the columns where the initial living expenses are $90,000 or above. The numbers in these columns are irrelevant, since the higher the annuity cost is, the lower the solvency probability is and the lower the expected bequest is. That is, these large increases merely represent the final effect of the bequest clause in cases where the investor has a high chance of becoming bankrupt due to purchasing annuities that should be completely avoided.

The relevant columns of Table 5 are where the initial living expenses are $80,000 or less, in which case the investor must balance their priorities. For example, if the investor has $80,000 of initial living expenses and wants to purchase an annuity for $250,000, the addition of the bequest clause reduces their solvency probability from 87.4% to 86.1%, but increases their expected bequest by $28,000 (from $799,000 to $827,000). This may or may not be worthwhile to the retiree. However, the table also indicates cases where the bequest clause clearly makes sense. For example, if the initial living expenses are $60,000 and the investor is intending to spend $1,250,000 on an annuity, adding the bequest clause provides an additional $10,000 to the expected bequest, while the solvency probability remains at 100.0%.

4.3.4 Changing The Expected Returns Or The Volatilities Of The Investment Portfolios

All of our previous results have relied on the investor having access in the non-annuitized portion of their portfolio to each of the 15 investments contained in Table 2. What happens if the values for the underlying expected returns, μ, or the volatilities, σ, given in Table 2 are incorrect? To gauge this effect, we stress test our results for the base cases by first increasing μ by two percentage points for each of the 15 investments, leaving the volatilities alone. This leads to the solvency-bequest frontier given in Table 6. Then we restore each μ to its Table 2 value, but increase each of the 15 values for σ in Table 2 by four percentage points. This leads to the solvency-bequest frontier given in Table 7.

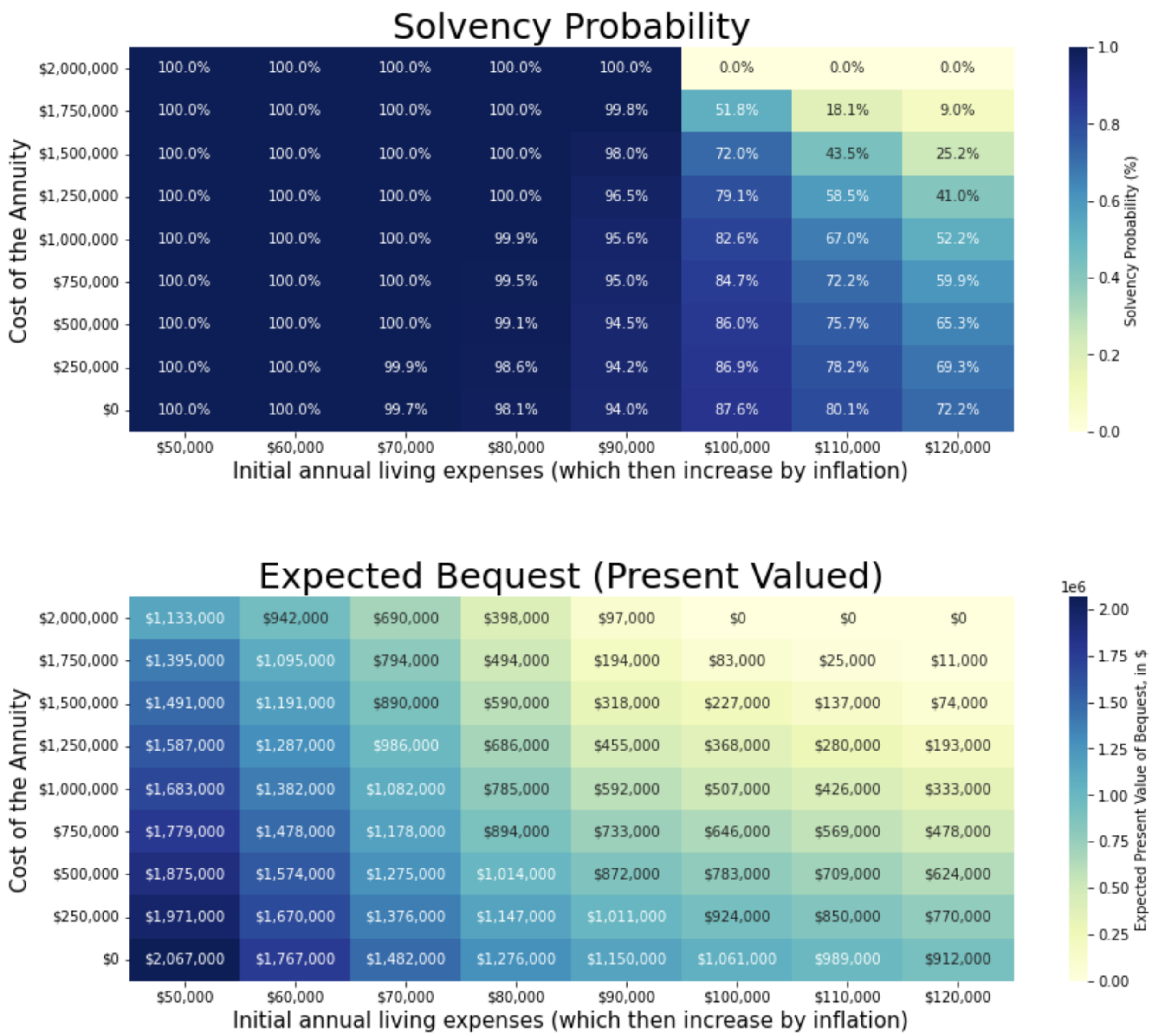
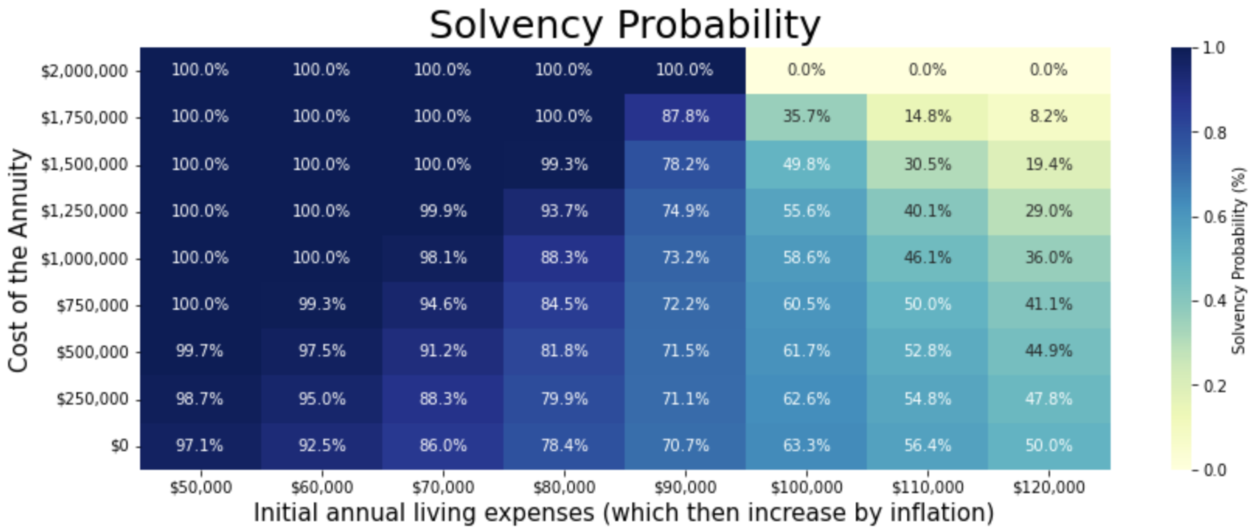
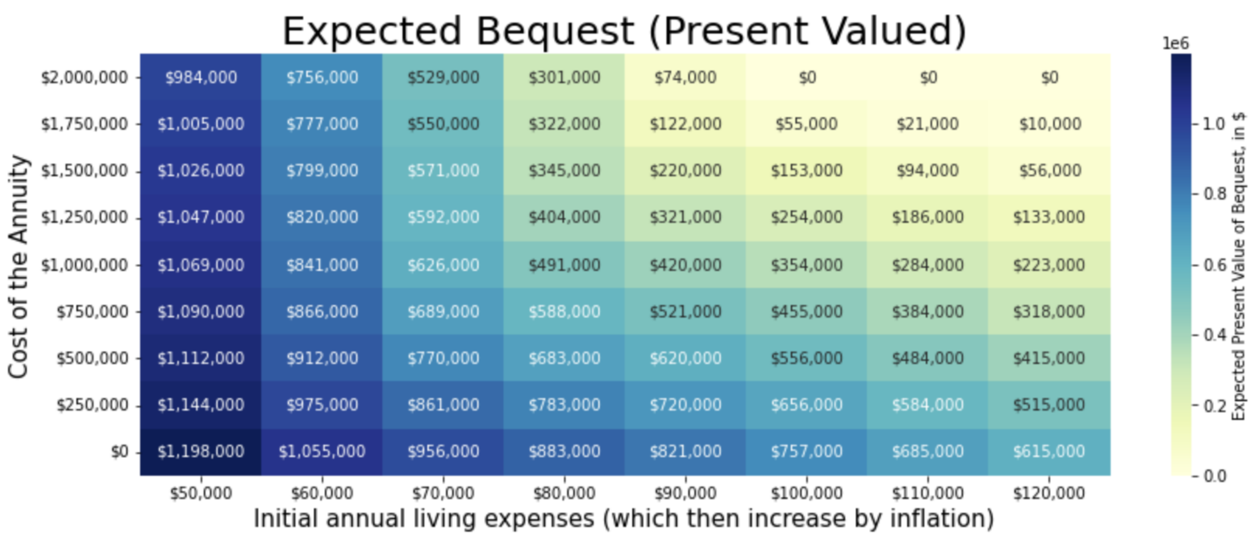


Table 6: The solvency-bequest frontier when each of the values for μ in Table 2 are increased by two percentage points.



Solvency Probability

Cost of the Annuity	$50,000	$60,000	$70,000	$80,000	$90,000	$100,000	$110,000	$120,000
$2,000,000	100.0%	100.0%	100.0%	100.0%	100.0%	0.0%	0.0%	0.0%
$1,750,000	100.0%	100.0%	100.0%	100.0%	87.8%	35.7%	14.8%	8.2%
$1,500,000	100.0%	100.0%	100.0%	99.3%	78.2%	49.8%	30.5%	19.4%
$1,250,000	100.0%	100.0%	99.9%	93.7%	74.9%	55.6%	40.1%	29.0%
$1,000,000	100.0%	100.0%	98.1%	88.3%	73.2%	58.6%	46.1%	36.0%
$750,000	100.0%	99.3%	94.6%	84.5%	72.2%	60.5%	50.0%	41.1%
$500,000	99.7%	97.5%	91.2%	81.8%	71.5%	61.7%	52.8%	44.9%
$250,000	98.7%	95.0%	88.3%	79.9%	71.1%	62.6%	54.8%	47.8%
$0	97.1%	92.5%	86.0%	78.4%	70.7%	63.3%	56.4%	50.0%



Expected Bequest (Present Valued)

Cost of the Annuity	$50,000	$60,000	$70,000	$80,000	$90,000	$100,000	$110,000	$120,000
$2,000,000	$984,000	$756,000	$529,000	$301,000	$74,000	$0	$0	$0
$1,750,000	$1,005,000	$777,000	$550,000	$322,000	$122,000	$55,000	$21,000	$10,000
$1,500,000	$1,026,000	$799,000	$571,000	$345,000	$220,000	$153,000	$94,000	$56,000
$1,250,000	$1,047,000	$820,000	$592,000	$404,000	$321,000	$254,000	$186,000	$133,000
$1,000,000	$1,069,000	$841,000	$626,000	$491,000	$420,000	$354,000	$284,000	$223,000
$750,000	$1,090,000	$866,000	$689,000	$588,000	$521,000	$455,000	$384,000	$318,000
$500,000	$1,112,000	$912,000	$770,000	$683,000	$620,000	$556,000	$484,000	$415,000
$250,000	$1,144,000	$975,000	$861,000	$783,000	$720,000	$656,000	$584,000	$515,000
$0	$1,198,000	$1,055,000	$956,000	$883,000	$821,000	$757,000	$685,000	$615,000

Table 7: The solvency-bequest frontier when each of the values for σ in Table 2 are increased by four percentage points.

Unsurprisingly, when we increase the μ values of the investments, both the solvency probability and the expected bequest increase. Comparing Table 6 to the base cases in Table 3 quantifies the sensitivity of the these increases.

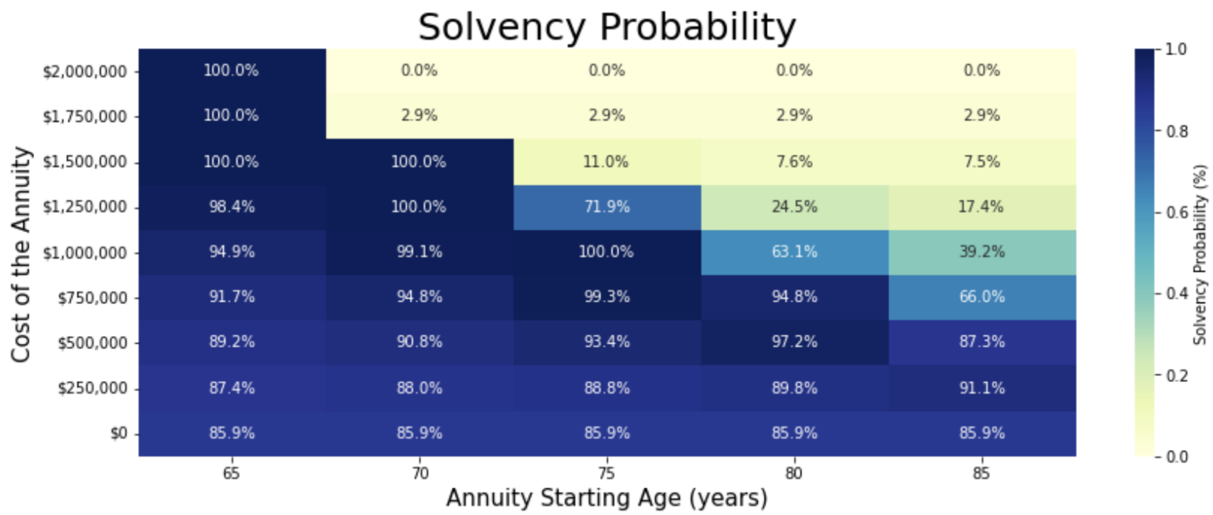
The effect of increasing the σ values, given by comparing Table 7 to Table 3, is more complex. We start with the solvency probabilities given in the top panels of the two tables. When the solvency probability is small, having additional volatility increases the probability of unusually good returns that allow the retiree to remain solvent until they pass away. So, for example, when the cost of the annuity is $1,500,000 and the initial annual living expenses are $120,000, the solvency probability increases from 18.5% in the base case to 19.4% when the volatilities increase by four percentage points. On the other hand, when the solvency probability is larger, the additional volatility decreases the probability of remaining solvent, since it increases the chance of unusually poor returns. For example, if there is no annuity and the retiree has initial living expenses of $70,000, then the additional volatility decreases the solvency probability from 93.0% to 86.0%. At what solvency probability does the additional volatility make no difference? We can see that when the cost of the annuity is \$1,500,000 and the initial annual living expenses are \$110,000, the solvency probability is 30.5%, with or without the additional volatility. For any case in the chart where the solvency probability is above 30.5%, the additional volatility decreases the solvency probability, and for any case in the chart where the solvency probability is below 30.5%, the additional volatility increases the solvency probability.

What is the effect of the additional volatility on the expected bequest shown in the bottom panels of Tables 7 and 3? There are two underlying effects: one is due to the change to the solvency probability. When the solvency probability goes down, it decreases the expected bequest, since bankrupt portfolios contribute nothing to the expected bequest. The other effect is that the solvent portfolios, on average, are bigger due to the nature of compounding. That is, if we have a portfolio worth \$100 and average the two cases of a 5% loss versus a 5% gain in each of 10 years, we have an expected portfolio worth of $\$100 \times (.95^{10} + 1.05^{10})/2 = \$111.38$, but if we consider a 10%, instead of 5%, loss or gain, the expected portfolio worth increases to $\$100 \times (.90^{10} + 1.10^{10})/2 = \$147.12$. In the cases where the additional volatility either increases the solvency probability or changes it very little (e.g., it stays at or close to 100%), we see the expected bequest increase due to the first effect either ameliorating or not outweighing the second effect. In the cases where the additional volatility decreases the solvency probability significantly enough, however, the first effect can outweigh the second effect. For example, in the case we discussed in the previous paragraph with no annuity and initial living expenses of \$70,000, the probability solvency decreased significantly, from 93.0% to 86.0%, and we see the expected bequest decrease from \$965,000 to \$956,000.

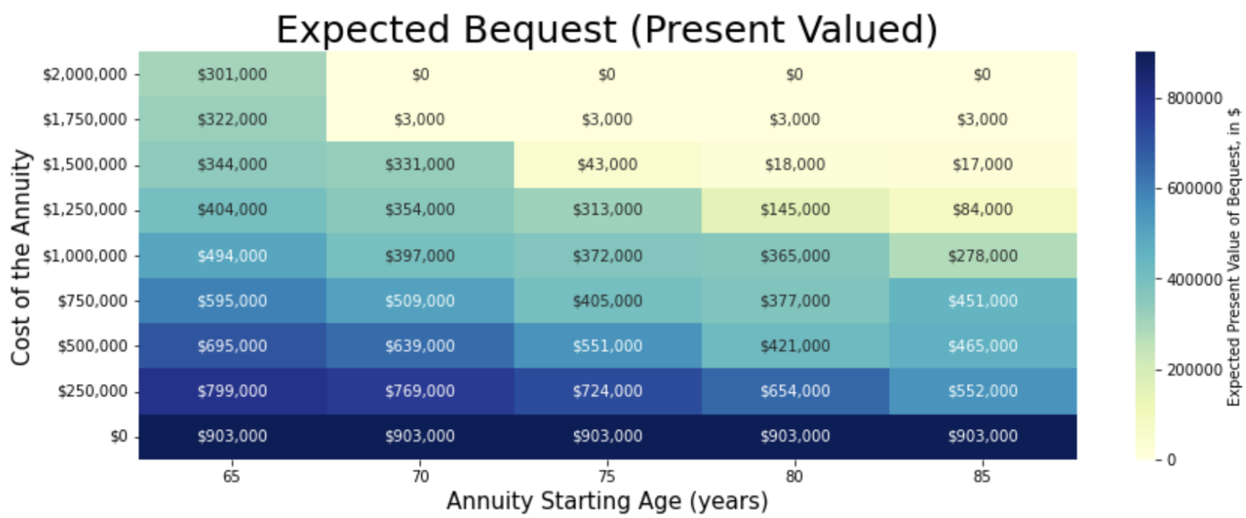
4.3.5 Deferred Payouts (DIAs)

What happens if instead of assuming the annuities' payouts begin immediately for the 65 year old retiree, as in the base cases, we allow the option of purchasing annuities whose initial payouts are deferred until the retiree reaches age 70 or 75 or 80 or 85? The corresponding initial annuity payouts are given in the 2nd through 5th columns in the top panel of Table 1. If we restrict ourselves to the case where the initial annual living expenses are \$80,000, the resulting solvency-bequest frontier is given in Table 8.

Deferring the annuity is generally worthwhile if there is enough money in the non-annuitized portion of the portfolio that its dynamically optimized investments have a high probability of keeping the retiree solvent until the annuity payouts start. This means the smaller the cost of the annuity, the longer it makes sense to defer payouts, since the smaller annuity increases the size of the non-annuitized portion of the portfolio, making it last longer.



Cost of the Annuity	65	70	75	80	85
$2,000,000	100.0%	0.0%	0.0%	0.0%	0.0%
$1,750,000	100.0%	2.9%	2.9%	2.9%	2.9%
$1,500,000	100.0%	100.0%	11.0%	7.6%	7.5%
$1,250,000	98.4%	100.0%	71.9%	24.5%	17.4%
$1,000,000	94.9%	99.1%	100.0%	63.1%	39.2%
$750,000	91.7%	94.8%	99.3%	94.8%	66.0%
$500,000	89.2%	90.8%	93.4%	97.2%	87.3%
$250,000	87.4%	88.0%	88.8%	89.8%	91.1%
$0	85.9%	85.9%	85.9%	85.9%	85.9%



Cost of the Annuity	65	70	75	80	85
$2,000,000	$301,000	$0	$0	$0	$0
$1,750,000	$322,000	$3,000	$3,000	$3,000	$3,000
$1,500,000	$344,000	$331,000	$43,000	$18,000	$17,000
$1,250,000	$404,000	$354,000	$313,000	$145,000	$84,000
$1,000,000	$494,000	$397,000	$372,000	$365,000	$278,000
$750,000	$595,000	$509,000	$405,000	$377,000	$451,000
$500,000	$695,000	$639,000	$551,000	$421,000	$465,000
$250,000	$799,000	$769,000	$724,000	$654,000	$552,000
$0	$903,000	$903,000	$903,000	$903,000	$903,000

Table 8: The effect of deferring the start of annuity payments. Note that unlike the previous five tables, the horizontal axis is now the annuity starting age, not the initial living expenses. The initial living expenses are assumed to be $80,000 throughout both panels in this table.

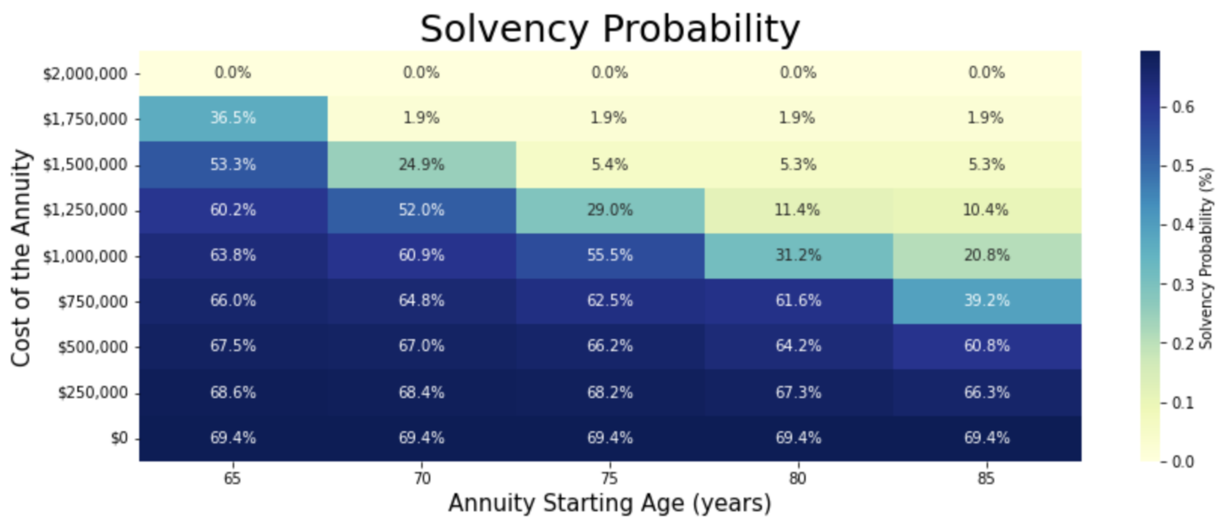
This effect can clearly be seen in the top panel of Table 8. If the cost of the annuity is $2,000,000, it leaves nothing for the non-annuitized portion of the portfolio, meaning any deferral of the annuity leads to immediate bankruptcy, which is reflected in the top line of the top panel. If the annuitized and non-annuitized portions of the portfolio are both $1,000,000, we see that the solvency probability is maximized (and 100%) by deferring the annuity until the age of 75. This makes sense since the non-annuitized $1,000,000 can address the first 10 years of annual living expenses, which range from $80,000 to $\$80{,}000 \times (1.02)^9 = \$95{,}607$, and then the annuity pays $105,185, which is bigger than $\$80{,}000 \times (1.02)^{10} = \$97{,}520$, the next year's living expenses. In addition, since both the annuity payouts and the living expenses increase by 2% each year, all future living expenses will be completely covered by the annuity. Further, of all the annuities that yield a 100% solvency probability, this is the best choice, since it has the highest expected bequest, as we can see from the bottom panel of Table 8.

It is important to be aware that the solvency probability panel hides an important factor that must be taken into account with deferred annuities, which is that the solvency probability tells us nothing about *when* bankruptcies will occur. With deferred annuities, bankruptcy is more likely to occur before the annuity payouts start. So while an investor intending to spend \$250,000 on an annuity could say they were wiser to defer their payouts until they were 85 years old because their solvency probability increased from 87.4% with an immediate payout annuity to 91.1%, they would be failing to take into account the fact that bankruptcies with the immediate annuity are much more likely to happen later in life, while with the deferred annuity, they are more likely to happen before the age of 85.

The top panel in Table 8 also shows that our rule for immediate annuities — namely, that a retiree who seeks to optimize their solvency probability should annuitize as much as possible if they are able to annuitize all their consumption needs, but should completely eschew annuities otherwise — does not hold for deferred annuities. Observing each of the columns in the panel, we see that the optimal fraction of the portfolio that should be annuitized changes as we change the age at which the annuity payouts begin. For example, if we intend to defer the annuity payouts until the age of 85, we are best off only spending \$250,000 on the annuity, which corresponds to a 91.1% solvency probability, the highest solvency probability in the age 85 column. Of course, restricting consideration of annuities only to those whose payouts start at age 85 can be a mistake. Indeed, it is a mistake here. It would be better to choose an annuity that cost \$750,000 and started paying out immediately, since that leads to a higher solvency probability (91.7% versus 91.1%) with, on average, later bankruptcies, and a higher expected bequest (\$595,000 versus \$552,000).

In each row of the lower panel in Table 8, we generally see that the expected bequest decreases as we defer the start of the annuity payouts to later. This is because the early annuity payouts enable the investor to take a longer term view of the non-annuitized portion of their portfolio, which corresponds to more aggressive investment portfolios and therefore, as we have shown, higher expected bequests. The exception is when the cost of the annuity is either \$500,000 or \$750,000, in which case the expected bequest increases when the annuity start age increases from 80 to 85. This is because the payout amounts starting at age 80 are generally used to address living expenses, while at age 85 the payout amounts almost double, quickly amassing a significant nest egg that is not needed to address living expenses. Nevertheless, these annuities that start payouts at age 85 and cost \$500,000 or \$750,000 are inferior to the one costing \$250,000, since that has both a higher solvency probability and a higher expected bequest.

Finally, we consider the effect of deferring the annuity payouts on a case where the living expenses are high enough that we know the immediate annuity should be avoided from our base cases analysis in Subsection 4.3.1. In Table 9, we consider what happens when the initial living expenses are \$100,000, where the immediate annuity reduced solvency probability, as opposed to \$80,000, where the immediate annuity increased solvency probability. From the upper panel in the table, we see that deferring the annuity only makes matters worse: that is, the more the annuity payments are deferred, the lower the solvency probability is. This means that for the cases we saw before where the retiree was best off avoiding immediate annuities, they should also avoid deferred annuities.



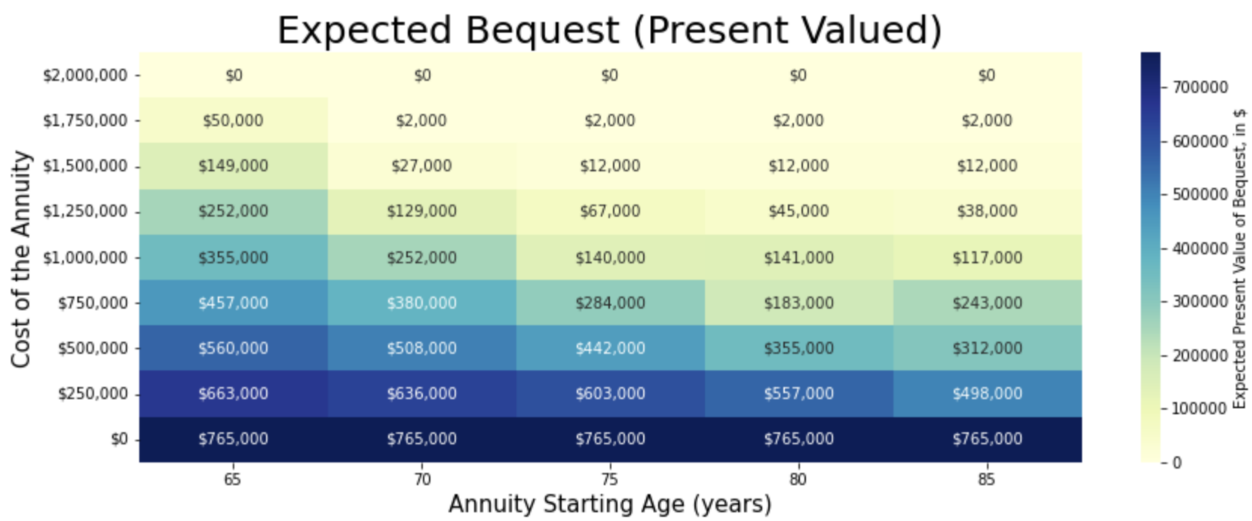


Table 9: The effect of deferring the start of annuity payments when the initial yearly living expenses are $100,000. From the top panel, we see that deferring the start of the annuity payments reduces the solvency probability. Worse, from the bottom panel, we also see that deferring the start of the annuity payments also generally reduces the expected bequest.

5 Concluding Comments

This paper considers a simple model for determining what, if any, annuity is most appropriate for a retiree, given their need to balance longevity risk, legacy risk, and lifestyle risk. It is assumed that investors who are interested in annuities care more about reducing longevity risk than legacy risk. Given this, we allow the investor to split their portfolio into annuitized and non-annuitized portions, but then dynamically optimize the investments in the non-annuitized portion of the portfolio to minimize longevity risk. We determine this minimized longevity risk (i.e., maximized solvency probability) and also the corresponding legacy risk (measured by the expected present value of the bequest).

We then investigated the effects of selecting various annuities on both the solvency probability and the expected present value of the bequest, given a variety of lifestyle risks (i.e., consumption rates/living expenses). Our main findings are:

1. The retiree should first look at SPIAs, that is, annuities whose payouts start immediately. If they are not able to afford annuitizing all of their consumption needs with an SPIA annuity, they should avoid any annuities. That is, the more they annuitize, the lower their solvency probability is. These findings are robust across varying expected capital market conditions, SPIAs with or without inflation adjustments, and annuities with or without bequest clauses. Deferring the start of payouts (DIAs) only reduces solvency probability further.

2. If the retiree is able to afford annuitizing all of their consumption, then they are best off doing so, since their primary goal is to optimize solvency probability. If they are legally prevented from completely annuitizing their consumption, then they are best off annuitizing as much as they can. However, the more they annuitize, the lower their expected bequest will be, which we quantify. Further, retirees may be able to maintain a nearly 100% chance of lifetime solvency without annuitizing all of their consumption, in which case they may prefer this lower level of annuitization since it will increase their expected bequest.

   These retirees may also wish to consider annuities with different inflation rates for their payouts, or with bequest clauses, or, in particular, deferred payout annuities (DIAs). These choices may give a more desirable solvency probability versus expected bequest payout or, in some cases, increase both the solvency probability and the expected bequest payout. Our results quantify this process, allowing each individual investor to knowledgeably make an annuity selection that is optimal for them.

The annuities in this paper cover standard, single-payment, fixed annuities with or without (a) inflation adjustments, (b) deferred payout starts, and (c) bequest refund clauses. However, the universe of annuities is large and may include other special features, such as stochastic payments indexed to macro-economic or market state variables, varied tax treatments, minimum withdrawal benefits, and guaranteed lifetime withdrawals, all of which inject option-like features or make the analysis path dependent. In future work, we can investigate the implications of features like these through different approaches, such as reinforcement learning, while still optimizing the investment strategy in the non-annuitized portion of the portfolio.

A Appendix: The Dynamic Programming Algorithm

The dynamic programming section of our algorithm determines the optimal investment strategy at a given wealth, W, and time, t. We note that W is the post-tax worth of all the investor's accounts, except, of course, for the annuity.

We consider a variety of possible investment portfolios indexed by l. These may be chosen, for example, to lie on the efficient frontier, though this is not required. Define μ_l and σ_l to be the mean and volatility of investment portfolio l. Let W_i be a grid of potential wealth values, W, indexed by $i = 0, 1, \ldots, i_{\max}$, where $W_0 = 0$ represents bankruptcy, W_1 is a very small amount, and the logarithm of the W_i are equally spaced as i increases from 1 to $i_{\max}$. We consider times $t = 0, 1, \ldots, T$, where t is assumed to be in years, although it can be other units of time, and $t = T$ corresponds to a time when the probability that the investor is still alive is extremely small, such as T corresponding to the investor being 120 years old, as we use for our examples in this paper.

The state space is a rectangular grid of (W_i, t) points over the range of i and t. Our goal is to find the optimal value of l at each point in the state space. From the optimal l, we can determine the value function $V(W_i, t)$, which is the optimal probability of remaining solvent if the investor is worth W_i at time t. It also gives the optimal investment portfolio strategy, which is simply to have the investor annually switch their investment portfolio to the optimal investment portfolio l associated to the closest W_i to the investor's wealth and the current time t.

We begin with the small assumption that the few investors that have not already died or become insolvent by $t = T$ will die at $t = T$, which implies the final time condition $V(W_i, T) = 1$ for all $i > 0$ (and $V(W_0, T) = 0$, since the investor is bankrupt in this case). We then iteratively evolve the value function backwards in time using three stages for each year. In Stage 1, we consider the possibility of death during the year, which, for simplicity, we address only here, at the end of the year. In Stage 2, we consider the evolution of the investment portfolio during the year, which is where we determine the optimal l. In Stage 3, we consider the consumption, which, again for simplicity, we address only here, at the beginning of the year. We next detail each of these three stages more concretely:

Stage 1: Mortality. Let $p_M(t)$ be the probability, as determined by the investor's mortality table, that an investor dies during year t, conditioned on their being alive at the beginning of the year. Since, for our model, we combine these deaths so they occur at the end of year t, the first intermediate value function, $V_1(W_i, t)$, just prior to these end of the year deaths is

$$V_1(W_i, t) = \begin{cases} 1 \times p_M(t) + V(W_i, t+1) \times (1 - p_M(t)) & \text{if } i > 0 \\ 0 & \text{if } i = 0. \end{cases}$$

This simply reflects the fact that should the investor die with $W > 0$, they remained solvent throughout their lifetime. We note that W does not reflect any money gained from the annuity after an investor becomes bankrupt. That is, once an investor becomes bankrupt, $W = 0$ in each subsequent year.

Stage 2: Investment evolution. We now look at the probability that a portfolio worth W_i at the beginning of year t will transition to being worth W_j at the year's end if it is in portfolio l. For this paper, we assume geometric Brownian motion, although any other Markovian evolution model can be used instead if desired. For geometric Brownian motion, the probability density, f, for transitioning from a given wealth grid point W_i if $i > 0$ to wealth grid point W_j if $j > 0$ is

$$f(W_j | W_i, l) = \phi\left(\frac{1}{\sigma_l}\left(\ln\left(\frac{W_j}{W_i}\right) - \left(\mu_l - \frac{\sigma_l^2}{2}\right)\right)\right),$$

where $\phi(z)$ is the value of the probability density function of the standard normal random variable at $Z = z$. If $i = 0$ and $j > 0$ or vice versa, then $f(W_j | W_i, l) = 0$. Finally, $f(W_0 | W_0, l) = 1$, since an investor that starts bankrupt stays bankrupt. Normalizing these probability density function values yields the desired transition probabilities:

$$p(W_j | W_i, l) = \frac{f(W_j | W_i, l)}{\sum_{k=0}^{i_{\max}} f(W_k | W_i, l)}.$$

This in turn gives the Bellman equation for the second intermediate value function $V_2(W_i, t)$:

$$V_2(W_i, t) = \max_l \left[\sum_{j=0}^{i_{\max}} V_1(W_j, t) p(W_j | W_i, l)\right].$$

The l that maximizes the right hand side in this Bellman equation is the optimal l we sought. It gives the optimal investment portfolio for the investor to select if they are worth W_i at this stage during year t.

Stage 3: Consumption. Let $C(t)$ represent any remaining consumption needs in year t not met by that year's annuity payout. We model this by taking out all of $C(t)$ at the very beginning of the year from the non-annuitized portion of the portfolio. Ideally, this would mean

$$V(W_i, t) = \begin{cases} 0 & \text{if } W_i - C(t) \leq 0 \\ V_2(W_i - C(t), t) & \text{if } W_i - C(t) > 0. \end{cases}$$

However, $W_i - C(t)$ is not generally a grid point and V_2 is only defined on grid points, so if $W_i - C(t) > 0$, we determine $V(W_i, t)$ by interpolating the values of V_2 at the two grid points just below and above $W_i - C(t)$. Should the annual annuity payout be greater than the annual consumption during year t, we just have $C(t) < 0$ and follow the same process.

References

Bellman, R. (1952, August). On the Theory of Dynamic Programming. Proceedings of the National Academy of Sciences 38(8), 716–719.

Benartzi, S., A. Previtero, and R. H. Thaler (2011). Annuitization Puzzles. Journal of Economic Perspectives 25(4), 143–164.

Bengen, W. (2004). Determining Withdrawal Rates Using Historical Data. Journal of Financial Planning (March), 171–180.

Blake, D. and W. Burrows (2001). Survivor Bonds: Helping to Hedge Mortality Risk. Journal of Risk and Insurance 68, 339–348.

Blanchett, D. (2013). Simple Formulas to Implement Complex Withdrawal Strategies. Journal of Financial Planning 26(9), 40–48.

Blanchett, D., M. Kowara, and P. Chen (2012). Optimal Withdrawal Strategy for Retirement Income Portfolios. The Retirement Management Journal 2(3), 7–20.

Cooley, P. L., C. M. Hubbard, and D. T. Walz (1998). Retirement Savings: Choosing a Withdrawal Rate That Is Sustainable. Journal of the American Association of Individual Investors 20(2), 16–21.

Das, S. R., D. Ostrov, A. Casanova, A. Radhakrishnan, and D. Srivastav (2021, July). Combining Investment and Tax Strategies for Optimizing Lifetime Solvency under Uncertain Returns and Mortality. Journal of Risk and Financial Management 14(7), 285. Number: 7 Publisher: Multidisciplinary Digital Publishing Institute.

Das, S. R., D. Ostrov, A. Radhakrishnan, and D. Srivastav (2020). Dynamic Portfolio Allocation in Goals-Based Wealth Management. Computational Management Science 17(June), 613–640.

Department of Labor (2019). Private Pension Plan Bulletin Historical Tables and Graphs 1975-2017, Table E7. Technical report, Employee Benefits Security Administration, United States Department Of Labor. Contract DOLOPS14D0017.

Finke, M., W. D. Pfau, and D. Blanchett (2013). The 4 Percent Rule Is Not Safe in a Low-Yield World. Journal of Financial Planning (June).

Finke, M. S. and D. Blanchett (2016). An Overview of Retirement Income Strategies. Journal of Investment Consulting 17(1), 22–30.

Friedman, B. M. and M. Warshawsky (1985a, August). The Cost of Annuities: Implications for Saving Behavior and Bequests. Technical Report 1682, National Bureau of Economic Research, Inc. Publication Title: NBER Working Papers.

Friedman, B. M. and M. J. Warshawsky (1985b, August). Annuity Prices and Saving Behavior in the United States. SSRN Scholarly Paper ID 338764, Social Science Research Network, Rochester, NY.

Guillemette, M. (2017). Risks in Advanced Age: A Review of Research and Possible Solutions. Journal of Financial Planning (September).

Horneff, W. J., R. Maurer, O. S. Mitchell, and M. Z. Stamos (2007, August). Money in Motion: Dynamic Portfolio Choice in Retirement. SSRN Scholarly Paper ID 962720, Social Science Research Network, Rochester, NY.

Hu, W.-Y. and J. S. Scott (2007, November). Behavioral Obstacles in the Annuity Market. Financial Analysts Journal 63(6), 71–82.

Huang, H., M. A. Milevsky, and T. S. Salisbury (2017). Retirement spending and biological age. Journal of Economic Dynamics and Control 84(C), 58–76. Publisher: Elsevier.

Huang, H., M. A. Milevsky, and V. R. Young (2017, March). Optimal Purchasing of Deferred Income Annuities When Payout Yields are Mean-Reverting*. Review of Finance 21(1), 327–361.

Kitces, M. and W. D. Pfau (2013, July). The True Impact of Immediate Annuities on Retirement Sustainability: A Total Wealth Perspective. SSRN Scholarly Paper ID 2296867, Social Science Research Network, Rochester, NY.

Martellini, L., V. Milhau, and J. Mulvey (2019, June). "Flexicure" Retirement Solutions: A Part of the Answer to the Pension Crisis? The Journal of Portfolio Management 45(5), 136–151. Publisher: Institutional Investor Journals Umbrella.

Martellini, L., V. Milhau, and J. Mulvey (2020, April). Securing Replacement Income with Goal-Based Retirement Investing Strategies. The Journal of Retirement 7(4), 8–26. Publisher: Institutional Investor Journals Umbrella.

Merton, R. C. and A. Muralidhar (2017, April). Time for Retirement 'SeLFIES'? SSRN Scholarly Paper ID 2945668, Social Science Research Network, Rochester, NY.

Milevsky, M. A. (1998). Optimal Asset Allocation towards the End of the Life Cycle: To Annuitize or Not to Annuitize? The Journal of Risk and Insurance 65(3), 401–426. Publisher: [American Risk and Insurance Association, Wiley].

Milevsky, M. A. (2013). Life Annuities: An Optimal Product for Retirement Income, Volume 1. CFA Institute.

Milevsky, M. A. and V. R. Young (2001, November). The Real Option to Delay Annuitization: It's Not Now-or-Never. SSRN Scholarly Paper ID 289548, Social Science Research Network, Rochester, NY.

Ngai, A. and M. Sherris (2011). Longevity risk management for life and variable annuities: The effectiveness of static hedging using longevity bonds and derivatives. Insurance: Mathematics and Economics 49(1), 100–114. Publisher: Elsevier.

Richards, S., G. L. Jones, and Staple Inn Actuarial Society (Eds.) (2004). Financial aspects of longevity risk: presented to the Staple Inn Actuarial Society, Staple Inn, London, 26th October 2004 6pm. London: Prudential Assurance Company. OCLC: 315085275.

Yaari, M. E. (1965, April). Uncertain Lifetime, Life Insurance, and the Theory of the Consumer1. The Review of Economic Studies 32(2), 137–150.