

Local Volatility and the Recovery Rate of Credit Default Swaps

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Abstract

Credit default swap (CDS) spreads can only be decomposed into the probability of default and the loss-given-default by imposing some structure. Employing a hybrid binomial tree for equities and a recovery function, Das and Hanouna (2009) obtain accurate estimates for CDS spreads by fitting the model to historical equity volatilities. We extend their approach by including the full implied volatility surface, developing an implied binomial tree with a jump to default based on extending the Derman and Kani (1994) tree. We then evaluate the effect of including the full volatility surface on the implied CDS recovery rate.

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1. Introduction

The yield on a corporate bond can be disentangled into two components: (1) the yield of a risk-free asset with the same maturity and (2) the spread over this risk-free asset based on the issuer's credibility. If the risk-free rate is the London interbank offered rate (LIBOR), then the credit spread is the asset swap spread (ASW) or the credit default swap spread (CDS). This credit spread is the compensation an investor requires for bearing the default exposure associated with the issuer. In a risk-neutral world, this credit spread should be equal¹ to the product of the probability that the issuer defaults (PD) and the loss the investor incurs in the case of a default (loss given default or LGD). This pivotal relationship is referred to as the "credit triangle."

First, the confounding aspect of this relationship is that although the credit spread is known in the form of a market price (ASW or CDS), it is the result of a product of the PD and LGD, which are therefore not separately identifiable. Additionally, the credit spread might include other components that are not related to default risk such as liquidity risk and any embedded options. If we abstract from this non-default component and want to decompose the credit triangle, we need some structure on either the PD or the LGD to identify the other component. Most research addresses the identification of either the PD or the LGD alone, assuming the other is a known constant. The modelling of default probabilities alone for example goes back to Altman (1968), where he composed an indicator (Z-score) to predict defaults based on a combination of accounting variables. The modelling of the LGD alone focuses on the determinants of actual defaults (see, e.g., Altman et al., 2005). The simplest solution, of course, is to assume a constant recovery rate, as practitioners often do by often assuming 40% as an estimate for the recovery rate on senior loans; however, this approach fails to take into account that LGD can be time varying: when a company defaults in a good economic environment, the recovery rate may be high, while if the company defaults in a recession, the assets might be sold for fire sale prices (see, e.g., Altman and Kuehne, 2012).

Second, another drawback for these approaches is that they are based on the physical world PDs and LGDs while the credit triangle relationship is based on risk-neutral PDs and LGDs. Bakshi et

¹ Under the assumption of constant hazard rates.

al. (2006) show that the risk-neutral PDs are higher than their physical counterpart, but also (not surprisingly given the credit triangle) that the risk-neutral LGDs are lower than their physical counterpart. Therefore, mixing physical world and risk-neutral variables can be complicated.

Extant research attempts to estimate the joint identification of the LGD and PD for sovereign and corporate CDS spreads. Merrick (2001) assumes a linear relationship for default to estimate the recovery rate for Argentina. Using a reduced form model, Zhang (2003) tests the model on Argentina as well. Pan and Singleton (2008) estimate recovery rates for sovereign countries using time-series data assuming a constant recovery rate and Elkamhi, Jacobs, and Pan (2014) estimate the cross-section of recovery rates of corporates bonds using time-series in a three-factor model. Doshi, Elkamhi, and Ornathanalai (2018) use bonds from the same issuer with different seniority (i.e. senior versus subordinated) to identify recovery rates. Using a similar approach, Schläfer and Uhrig-Homburg (2014) investigate different seniority of CDS premia (CDS versus LCDS) to examine the recovery rates assuming the same default likelihood. In contrast to these models, our approach only uses cross-sectional volatility observations from a single point in time to estimate recovery rates. Because it is parsimonious and forward-looking, and fits the current prices of CDS *and* options, our methodology is ideally suited for real-world application, i.e., extracting PDs, LGDs, and pricing CDS and options.

Berd (2005) demonstrates that it is possible to disentangle the credit triangle using CDS and digital default swaps. Das and Hanouna (2009) develop a model that identifies the PD and LGD jointly with information from the CDS and equity markets. Augmenting the binomial tree of Cox, Ross, and Rubinstein (1979) with a jump to default, they calibrated their model with CDS prices in order to estimate the term structure of PD and LGD. The advantage of the Das-Hanouna (hereafter DH) model is that it assumes no explicit correlation between the default likelihood and recovery rates, and does not require traded digital default swaps. However, as DH note, one deficiency of their model is that it does not use the full options volatility surface. The model only uses the at-the-money volatility and therefore ignores the fact that out-of-the money put options trade at a higher implied volatility. To include the full volatility surface, the Cox-Rubenstein-Ross (CRR hereafter) tree needs to be replaced by an implied tree (Derman and Kani, 1994 and Barle and Cakici, 1998) and then

augmented with a jump to default. DH state that they leave this extension for further research because it will enhance complexity but note that it is an important extension as it will use the entire volatility surface and raise the model's information content.

DH combine information from the options market and the credit market to estimate recovery rates. Carr and Wu (2011) also show that these markets are closely related by the simple link between a stock's deep-out-of-the money put option and its CDS spread. Therefore, no arbitrage requires that the DH model is calibrated to option implied volatility. Linking the CDS market with the options market assumes perfect integration of these two markets. This assumption seems strong as Carr and Wu (2011) show that there are cross market deviations. However these deviations go in both directions, indicating that no one of the two markets is leading the other all the time. Also, Culp, Nozawa, and Veronesi (2017) show that credit spreads from their pseudo firms (that are based on equity options) are close to observed corporate bond spreads. Therefore, Culp, Nozawa, and Veronesi (2017) suggest *“that there is a good deal of integration between corporate bond and (equity) markets”*.

In this paper, we extend the DH model to include the full volatility surface and then look at the effect on the recovery rate. Derman and Kani (1994), DK hereafter, develop an implied binomial tree that can exactly price options: each node in the tree is implied from the prices of traded options. To our knowledge, the augmentation of an implied tree with a jump to default, as undertaken here, is new. We are only aware of an extension to a trinomial tree (Derman and Chriss, 1996) where the stock price can either go up, down, or stay at the same level, or the jump-to-default model with constant elasticity of variance proposed by Carr and Linetsky (2006) and applied to the estimation of the recovery rate in a discrete framework by Le (2015). In a continuous-time framework, Carr and Madan (2010) develop a model for local volatility with a jump to default. In contrast to our model, their model does not calibrate directly on the recovery rate, but on an estimated survival function with a (fixed) recovery rate as one of the parameters. Moreover, because their model is in a continuous-time framework, it is more challenging to implement in contrast to a binomial tree structure that is easy to calibrate.

This paper contributes to the literature and practice in three important areas. First, we show that the DH model generates unrealistically high volatilities as a result of the jump to default. Therefore we need to control the volatility in the tree and develop a tree that can be calibrated on the option implied volatility surface (skew and term structure) to analyze the impact on the recovery rate.

Second our research contributes to the literature on the development of models for PD and LGD for asset and risk management applications, as well as regulatory requirements. Assuming a constant recovery rate biases credit value adjustment calculations (e.g., François and Jiang, 2016) under Basel 3 while IFRS9 requires banks not only to estimate the default probability but also to estimate the recovery rate in case of a default.

Third, because calibration makes the model consistent with the volatility surface, it can help in pricing longer-term options. The most liquid CDS have five-year tenors, while the option market only has maturities up to two years. So the model helps to answer whether market participants might benefit by using CDS contracts to price longer-term options.

The rest of the paper proceeds as follows. In Section 2, we review the DH model and establish notation. Section 3 develops the extended model to incorporate the entire volatility surface with a jump to default. Section 4 describes the data and Section 5 presents detailed empirical analyses using the model. This includes a comparison of four recovery rate models. We find that the best fit recovery rates come from models that focus on the term structure of implied volatilities rather than the skew. Concluding comments are offered in Section 6.

2. Volatility in the Das and Hanouna (DH) model

DH developed a model that uses information from equity and credit markets to extract implied recovery rates. The model is based on three building blocks (1) a binomial tree enhanced by a jump to default, (2) a CDS pricing model, and (3) a recovery function. In this section, we discuss how volatility is incorporated into the first building block of the DH model. The second and third building blocks will be discussed in Section 3.9.

The first building block is based on a standard CRR binomial tree, defined over a state space (i,j) , where time is indexed by j , and i indexes the state at time j . In this binomial tree, the initial stock price S_j^i in state i can either go up to state $i+1$ in the next period $j+1$, i.e., to S_{j+1}^{i+1} or down to state i , to value S_{j+1}^i . The stock price changes in the tree with a constant volatility: it goes up to $S_j^i e^{\sigma\sqrt{h}}$ or goes down to $S_j^i e^{-\sigma\sqrt{h}}$ in the next period. The movements are thus governed by two parameters, the volatility σ and the time step h . Note also that the binomial tree is a recombining tree, since as the stock price goes up in period $j+1$ and then down in period $j+2$ we arrive at the same price as at time j . The probability that the stock price goes up is q and therefore the probability that the stock price goes down is $(1 - q)$.

DH enhance the CRR tree with a jump to default. More specifically, in addition to an up and down movement as just described, the stock price S can also jump to default (drops to a price of zero) with probability δ . The company will survive if it does not default, therefore the probability of survival z is equal to $1 - \delta$. Below we schematically show the three movements as follows

$$S_j^i \longrightarrow \begin{cases} S_{j+1}^{i+1} = S_j^i e^{\sigma\sqrt{h}} & \text{Up state with probability } q_j^i z_j^i \\ S_{j+1}^i = S_j^i e^{-\sigma\sqrt{h}} & \text{Down state with probability } (1-q_j^i) z_j^i \\ 0 & \text{Default with probability } 1 - z_j^i \end{cases} \quad (1)$$

A requirement in a binomial tree is that the branching process is risk-neutral, meaning that the forward price of the stock should be equal to the product of the probabilities of the states and the stock prices of the different states.

Given equation (1) and the requirement of the risk-neutral branching process, the stock prices and probabilities should satisfy:

$$F_{j,j+1}^i = q_j^i z_j^i S_{j+1}^{i+1} + (1 - q_j^i) z_j^i S_{j+1}^i \quad (2)$$

with $F_{j,j+1}^i$ being the stock's forward price at time j for period $j+1$ and the additional requirement being that the transition probabilities (z_j^i and q_j^i) should lie in the interval $[0,1]$.

The unknown parameters in equation (2) are the probabilities z_j^i and q_j^i . Note also that z_j^i is state and time dependent, therefore if we have five layers in the binomial tree, we need for the first

four layers an estimate of z_j^i and therefore we need 10 survival probabilities to construct the tree.² To identify equation (2), DH assumed a functional form for z_j^i : the survival probability is a function of the stock price S . The structure is set up in two stages.

In the first stage, the default intensity ξ_j^i is modeled as a function of the stock price as follows:

$$\xi_j^i = \frac{1}{(S_j^i)^b} \quad (3)$$

where b is assumed to be greater than 0 so that default intensity decreases if the stock price rises. The stock price should act as a buffer as it does in structural models. The more buffer there is, the less is the default intensity.

In the second stage, the probability that the firm will survive between period j and $j+1$, z_j^i , will be an exponential function of the default intensity and time step t as follows:

$$z_j^i = e^{-\xi_j^i h} \quad (4)$$

Since ξ_j^i and h are always positive, the survival probability z_j^i will be between 0 and 1. In the limit, this means that the firm always survives ($z = 1$) or always defaults ($z = 0$).

The relation between b and z is positive: if b decreases, the survival probability decreases. The relationship between S and z is positive; the higher the stock price, the higher the survival probability.

The CRR binomial tree enhanced by a jump to default (hybrid tree) is estimated in DH with the parameter b . To analyze how the b parameter influences the volatility surface, we could theoretically calculate the local volatility at each node, but we choose to calculate implied volatilities directly from option prices because it is simpler and consistent with the way we calculate implied volatilities later.

The CRR binomial tree was originally developed as an easy alternative to understand the Black-Scholes formula (Black and Scholes, 1973). The method starts with creating a binomial tree for a call option with a given volatility and initial stock price. The last layer in the binomial tree gives the

² If n is the number of layers, we need $\frac{n(n+1)}{2}$ estimates.

stock prices at option's expiration date. Then the call option price is calculated by subtracting the exercise price from each terminal state price and multiplying it by the Arrow-Debreu state price (λ). The price of a call option with exercise price K and time to expiration j is:

$$C[K, j] = \sum_{i=0}^j \lambda_j^i \max(S_j^i - K, 0) \quad (5)$$

The Arrow-Debreu price (λ) is the price of a security that has a one unit payoff if the tree arrives at that state and a zero unit payoff otherwise. If there is no discounting, the Arrow-Debreu price is equal to the probability of reaching that state. Otherwise, the probability needs to be discounted using the risk-free rate of interest. The Arrow-Debreu price is thus in fact the discounted state probability.

When we have calculated the price of an option as explained above, we can plug the price and all the parameters (time to expiration, interest rate, stock price and strike) back into the Black-Scholes formula to solve for the implied volatility.³ The implied volatility is just another way of quoting the option's price. We can construct an implied volatility surface by varying (1) the exercise price (e.g., the log moneyness) and (2) the time to expiration.

The relationship between the moneyness and the implied volatility is flat in the CRR binomial tree, just as in the Black-Scholes formula. However, the relationship that has been observed in the options market is not flat, but has a skewed structure with a negative slope. Out-of-the-money put options (negative log moneyness) have higher implied volatilities than at-the-money options (zero log moneyness). This phenomenon is referred to as the "volatility skew".⁴

The relationship between time and implied volatility is flat in the CRR binomial tree, just as in the Black-Scholes formula. However, the relationship that has been observed in the options market is not flat, but either upward or downward sloping and this phenomenon is called "the term structure of volatility".

³ It may be a surprise, but the implied volatility is not exactly equal to the volatility we have used to construct the CRR binomial tree. One of the reasons is that the CRR method is only approximating the Black-Scholes formula asymptotically (e.g., the time step in the binomial tree should be very small).

⁴ Three patterns that are observed in the options market: (1) volatility smile (curvature around the at-the-money point-and prevalent in currency markets), (2) volatility skew (slope of the curve-and prevalent in equity markets), and (3) volatility smirk (superposition of the smile and the smirk-prevalent). See Zhang and Xiang (2008).

DH augmented the CRR binomial tree with an additional node so that the stock price can also jump to default. The risk of a default increases if either b becomes lower or the stock price becomes lower as a result of the increased default intensity in equation (3). As we know from the above, the CRR binomial tree will produce a flat volatility curve and the question we would like to answer is if the addition of the b parameter would change this. Therefore, we run a few scenarios, where we vary several parameters; that is, we investigate the impact of the parameter b , the moneyness (M), initial stock price (S), and the number of steps in the binomial tree on the implied volatility. In our experiment, we analyze the impact on a two-year option, setting interest rates and dividends equal to zero.⁵

<<Insert Figure 1 about here>>

The results of the experiment are shown in Figure 1. In panel a of the figure, we have set the time step in the binomial tree to 0.5 year (6 months)⁶ and volatility to 25%. Then we vary the b from 0.5 to 2 with steps of 0.5 and include a very high b of 100. If we set b to 100, the default intensity in equation (3) will be approximately zero and therefore the survival probability z is approximately one. Therefore, this case can be seen as the standard CRR binomial tree where there is no jump to default.

Two things are quite apparent from panel a in Figure 1: (1) when we increase b from the initial value of 0.5, the curve rapidly converges from a skewed curve to a hump shaped curve where there is no visible difference between a curve with a b value of 1.5 and a b value of 100; and (2) the curve converges, not to the straight line where $\sigma = 0.25$, but to a hump shaped curve.

Considering this last observation, we know that the binomial model for pricing options is only asymptotically an approximation for the Black-Scholes formula. Therefore, we increase the step size in the tree from 0.5 years to 0.05 years in panel b in Figure 1 and see that the implied volatility

⁵ We first calculate option prices for each exercise price by first calculating the payoff in the binomial tree and second multiplying this payoff by the Arrow-Debreu security price. Then we derive the implied volatility from the Black-Scholes model.

⁶ This means we will have five layers in the tree. The initial stock price at time $t=0$ and then then the layers at $t=0.5$, $t=1$, $t=1.5$ and $t=2$.

converges to a straight line as there is hardly any difference between the highest values of b and the straight line $\sigma = 0.25$ ⁷.

Based on the graphs in Figure 1, we can make two observations on the volatility structure in the DH model. First, the skew is only apparent for stocks with low values for b and/or low stock prices. As is shown in the study by Das and Hanouna (2009),⁸ the best three quintiles in terms of expected default frequency have a b value of around 1.25 and a stock price of around 20 should therefore, based on our analyses, have a rather flat volatility surface.

Second, when there is a skew, the at-the-money volatility (moneyness is zero) is much higher than the option market indicates. Das and Hanouna (2009) report that the worst quintile in terms of expected default frequency has a b value of 0.35. However we see in Figure 1 that a higher value of b not only skews the implied volatility curve, but also shifts the implied volatility upwards. This indicates that even the at-the-money volatility is not similar to the at-the-money option implied volatility.

For these reasons, we explore an extension of the DH model wherein we calibrate the binomial model based on information from options. These models, referred to as implied binomial trees, do not allow for a jump to default. Therefore we have to enhance these models with an additional jump to default as is done in DH.

The original idea of an implied binomial tree came from DK, who constructed a tree that is recombining and can exactly price options with different strike prices and thus different implied volatilities. Barle and Cakici (1998), BC hereafter, observed that the DK methodology can sometimes lead to negative transition probabilities and non-risk-neutrality at the branch level, especially when interest rates are high. They proposed two modifications: (1) setting the option's exercise price equal to the stock's forward price instead of the stock price and (2) centering the tree on the stock's forward price instead of the initial stock price. We included both modifications in our implied tree with a jump to default.

⁷ The scenarios that we ran where we varied the initial stock price from 100 to 10 and varied the initial volatility from 0.25% to 0.5% are available upon request from the authors.

⁸ See Table 2 in their study. Note, however, in the published article the b values are inadvertently omitted. They are available in the authors' working paper that is available on Social Science Research Network (<https://ssrn.com/abstract=1028612>).

3. Methodology

In this section, we discuss our methodology of how we can add a jump to default to an implied tree.

3.1 Implied tree with a jump to default

The world of implied trees is not as simple as the world of the CRR tree. In the CRR world, as can be seen in equation (1), if we know the volatility at time $t = 0$, the next nodes in the tree are given by the exponential of the volatility times the time step. In the world of implied trees, we do not know the value of S_{j+1}^i and S_{j+1}^{i+1} in equation (1) and need to construct the tree node by node in each layer by calibrating it to option prices

3.2 Risk Neutrality

The binomial trees of DK, BC, and CRR are based on the no-arbitrage condition. Therefore, the branching process in our implied tree with a jump to default must satisfy the risk-neutral condition for the branching process in equation (1) and will be equal to equation (2).

We can rewrite equation (2) in terms of probability q_j^i :

$$q_j^i = \frac{\frac{F_{j,j+1}^i}{z_j^i} - S_{j+1}^i}{S_{j+1}^{i+1} - S_{j+1}^i} \quad (6)$$

In the BC implied tree there is no default probability and therefore z_j^i equals one. If we do this in equation (6), then we arrive at no-arbitrage condition in the BC model. Based on that equation, as discussed at the end of Section 2, BC proposed two modifications to the original DK tree. We will apply the same two modifications to our implied tree with a jump to default model.

First we set the exercise price K_j^i equal to the forward rate $F_{j,j+1}^i$ divided by the survival probability z_j^i

$$K_j^i = \frac{F_{j,j+1}^i}{z_j^i} \quad (7)$$

Second, we let the tree grow with K_j^i . Therefore, the BC tree will be a nested solution of our model when the survival probability z_j^i is equal to one.

3.3 Arrow-Debreu Pricing

We introduced the Arrow-Debreu price (λ) in Section 2 as the discounted probability of reaching a certain node in a tree. The upper-most and lower-most nodes in a binomial tree are easy to describe in terms of Arrow-Debreu pricing since they can only be accessed from one previous node.

We can describe the Arrow-Debreu price for the upper most branch as:

$$\lambda_j^i = e^{-r_{j-1}h} \times \lambda_{j-1}^{i-1} \times q_{j-1}^{i-1} \times z_{j-1}^{i-1} \quad (8)$$

The Arrow-Debreu price λ_j^i is defined as the previous Arrow-Debreu price λ_{j-1}^{i-1} times the probability of the stock going up in the next period q_{j-1}^{i-1} times the probability that it does not default z_{j-1}^{i-1} and then we discount the value using $e^{-r_{j-1}h}$.

In a similar way we can define the Arrow-Debreu prices for the lowest nodes in the tree as:

$$\lambda_j^i = e^{-r_{j-1}h} \times \lambda_{j-1}^i \times (1 - q_{j-1}^i) \times z_{j-1}^i \quad (9)$$

where now $(1 - q_{j-1}^i)$ is the probability that the stock goes down.

For the nodes in between the upper and lower nodes (if any), we need to calculate the Arrow-Debreu prices from the two directions it can come from (i.e., from a node above and a node below the current level):

$$\lambda_j^i = e^{-r_{j-1}h} \times (\lambda_{j-1}^{i-1} \times q_{j-1}^{i-1} z_{j-1}^{i-1} + \lambda_{j-1}^i \times (1 - q_{j-1}^i) \times z_{j-1}^i) \quad (10)$$

3.4 Binomial Option Pricing

We can value an option with the Black-Scholes formula, but since we need to know the stock prices in the binomial tree, we apply binomial option pricing where the value of the option is equal to the value of the payoff of that option times the Arrow-Debreu security price in that state. So the price of a call option $C[K, j]$ with exercise price K and time to maturity j is equal to the payoff in layer j times the corresponding Arrow-Debreu state price λ_j^i as in equation (5).

3.5 The upper nodes in the implied tree with a jump to default

Now that we have all the ingredients (risk-neutrality, Arrow-Debreu prices, and binomial option pricing) we can start building the implied tree with a jump to default.

We know from equation (7) that the exercise price is equal to K_j^i . Therefore equation (5) becomes:

$$C[K_{j-1}^k, j] = \sum_{i=0}^j \lambda_j^i \max(S_j^i - K_{j-1}^k, 0) \quad (11)$$

The exercise price K_{j-1}^k is also chosen because it falls between the stock price when it goes up (S_j^{i+1}) and when it goes down (S_j^i) as probability q_{j-1}^i must be between zero and one. Therefore we will separate equation (11) in the counters that sum up to k and the ones above:

$$C[K_{j-1}^k, j] = \sum_{i=0}^k \lambda_j^i \max(S_j^i - K_{j-1}^k, 0) + \sum_{i=k+1}^j \lambda_j^i \max(S_j^i - K_{j-1}^k, 0) \quad (12)$$

We know that the payoff of the first summation part in equation (12) is always zero (as the exercise price is above the stock price) and the payoff in the second summation part is always positive (as the stock price is always above the exercise price). Therefore, equation (12) reduces to:

$$C[K_{j-1}^k, j] = \sum_{i=k+1}^j \lambda_j^i (S_j^i - K_{j-1}^k) \quad (13)$$

Then we separate both $i=j$ and $i=k+1$ from the summation term:

$$C[K_{j-1}^k, j] = \lambda_j^j (S_j^j - K_{j-1}^k) + \lambda_j^{k+1} (S_j^{k+1} - K_{j-1}^k) + \sum_{i=k+2}^{j-1} \lambda_j^i (S_j^i - K_{j-1}^k) \quad (14)$$

Remember from Section 3.3 on Arrow-Debreu pricing that the uppermost and lowermost branches are the simplest to describe as they can only be accessed from one previous node. We can use this to replace the λ 's (Arrow-Debreu prices) in equation (14) with the λ 's from equations (8) to (10):

$$\begin{aligned} C[K_{j-1}^k, j] = e^{-r_{j-1}h} & \left\{ \lambda_{j-1}^{j-1} q_{j-1}^{j-1} z_{j-1}^{j-1} (S_j^j - K_{j-1}^k) + (\lambda_{j-1}^k q_{j-1}^k z_{j-1}^k \right. \\ & + \lambda_{j-1}^{k+1} (1 - q_{j-1}^{k+1}) z_{j-1}^{k+1} (S_j^{k+1} - K_{j-1}^k) \\ & \left. + \sum_{i=k+2}^{j-1} (\lambda_{j-1}^{i-1} q_{j-1}^{i-1} z_{j-1}^{i-1} + \lambda_{j-1}^i z_{j-1}^i (1 - q_{j-1}^i)) (S_j^i - K_{j-1}^k) \right\} \end{aligned} \quad (15)$$

Then with some mathematical manipulation that we show stepwise in Appendix A, we arrive at:

$$S_j^{k+1} = \frac{S_j^k (C[K_{j-1}^k, j]e^{r_{j-1}h} - \rho_{up}) - \lambda_{j-1}^k z_{j-1}^k K_{j-1}^k (K_{j-1}^k - S_j^k)}{C[K_{j-1}^k, j]e^{r_{j-1}h} - \rho_{up} - \lambda_{j-1}^k z_{j-1}^k (K_{j-1}^k - S_j^k)} \quad (16)$$

where $\rho_{up} = \sum_{i=k+2}^j \lambda_{j-1}^{i-1} z_{j-1}^{i-1} \left(\frac{F_{j-1, j}^{i-1}}{z_{j-1}^{i-1}} - K_{j-1}^k \right)$

Equation (16) parameterizes S_j^{k+1} in terms of S_j^k and reduces to the solution of BC (See Barle and Cakici, 1998, equation (12)) when we set z equal to one.

3.5 The central nodes in the implied tree with a jump to default

When we start growing the tree, we do not know S_j^k and as a result cannot solve equation (16). Consequently, we have to make an assumption about the central nodes as was done by DK and BC. Following BC, we center the tree on the expected path. BC centered the tree on the stock's forward price using the risk-neutrality condition in equation (6) when z is equal to one; therefore we center the tree on the forward rate divided by the survival probability as in our model.

So when j is even⁹

$$S_j^{j/2} = \frac{F_{j-2, j}^{(j-2)/2}}{\left(\frac{j-2}{2} z_{j-2} \right)^2} \quad (17)$$

and we can use equation (17) in equation (16), so that the node above the central node is given by:

$$S_j^{k+1} = \frac{\frac{F_{j-2, j}^{(j-2)/2}}{\left(\frac{j-2}{2} z_{j-2} \right)^2} (C[K_{j-1}^k, j]e^{r_{j-1}h} - \rho_{up}) - \left(\lambda_{j-1}^k z_{j-1}^k K_{j-1}^k \left(K_{j-1}^k - \frac{F_{j-2, j}^{(j-2)/2}}{\left(\frac{j-2}{2} z_{j-2} \right)^2} \right) \right)}{C[K_{j-1}^k, j]e^{r_{j-1}h} - \rho_{up} - \lambda_{j-1}^k z_{j-1}^k (K_{j-1}^k - S_j^k)} \quad \forall j = \text{even} \quad (18)$$

When j is odd, we choose the following centering condition:

$$S_j^{(j+1)/2} = \frac{\left(\frac{F_{j-1, j}^{(j-1)/2}}{z_{j-1}^{(j-1)/2}} \right)^2}{(S_j^{(j-1)/2})} \quad (19)$$

⁹ We will make a small approximation here as we take $\left(\frac{z_{j-2}^{(j-2)/2}}{z_{j-1}^{(j-2)/2}} \right)^2$ instead of $\frac{z_{j-2}^{(j-2)/2}}{z_{j-1}^{(j-2)/2}}$. We take the survival probability at $j-2$ as this node has one central node instead of $j-1$ that has two central nodes. This is mathematically easier to implement.

This is similar to the approach used by CRR and BC where they center the tree on the exercise price.

In CRR, the centering condition is $S_j^{(j+1)/2} = \frac{(S_{j-1}^{(j-1)/2})^2}{(S_j^{(j-1)/2})}$ and in the BC model the centering condition

$$\text{is } S_j^{(j+1)/2} = \frac{(F_{j-1,j}^{(j-1)/2})^2}{(S_j^{(j-1)/2})}.$$

We can then replace the lower central node prices S_j^k in equation (16) with equation (19)

which takes the form of (e.g.: $k=(j-1)/2$ and $k+1=(j+1)/2$)

$$S_j^k = \frac{(K_{j-1}^k)^2}{S_j^{k+1}} \quad (20)$$

which results in:

$$S_j^{k+1} = \frac{\frac{(K_{j-1}^k)^2}{S_j^{k+1}} (C(K_{j-1}^k, j)e^{r_{j-1}h} - \rho_{up}) - \lambda_{j-1}^k z_{j-1}^{k,k+1} K_{j-1}^k \left(K_{j-1}^k - \frac{(K_{j-1}^k)^2}{S_j^{k+1}} \right)}{C[K_{j-1}^k, j]e^{r_{j-1}h} - \rho_{up} - \lambda_{j-1}^k z_{j-1}^k \left(K_{j-1}^k - \frac{(K_{j-1}^k)^2}{S_j^{k+1}} \right)} \quad (21)$$

Then after some mathematical manipulations, that we show stepwise in Appendix B, this becomes:

$$S_j^{k+1} = K_{j-1}^k \frac{\lambda_{j-1}^k F_{j-1,j}^k + (C[K_{j-1}^k, j]e^{r_{j-1}h} - \rho_{up})}{\lambda_{j-1}^k F_{j-1,j}^k - (C[K_{j-1}^k, j]e^{r_{j-1}h} - \rho_{up})} \quad \forall j = \text{odd} \ \& \ k = (j+1)/2 \quad (22)$$

which is our solution for the upper central node ($k=(j+1)/2$) and j is not even.

We can use the centralizing condition (20) to find the lower central node:

$$S_j^k = K_{j-1}^k \frac{\lambda_{j-1}^k F_{j-1,j}^k - (C[K_{j-1}^k, j]e^{r_{j-1}h} - \rho_{up})}{\lambda_{j-1}^k F_{j-1,j}^k + (C[K_{j-1}^k, j]e^{r_{j-1}h} - \rho_{up})} \quad \forall j = \text{odd} \ \& \ k = (j-1)/2 \quad (23)$$

Equation (23) reduces to the solution of BC (See Barle and Cakici, 1998, equation (13)) when we set z equal to one.

3.6 The lower nodes in the implied tree with a jump to default

In BC and DK the solution for the lower part of the tree is much simpler since we already know the upper nodes. In our case the price of the put option has another component that represents the price of a jump to default:

$$P(K_{j-1}^k, j) = \sum_{i=0}^j \lambda_j^i \max(K_{j-1}^k - S_j^i, 0) + \lambda_j^{jtd} \max(K_{j-1}^k - 0, 0) \quad (24)$$

Since we have assumed that the price of the stock jumps to zero in the case of a default, the last part can be simplified to just $\lambda_j^{jtd} K_{j-1}^k$. This last part is the price of a jump to default option that only pays off in case of a default and therefore has very similar characteristics as a CDS. We can express this jump to default option as:

$$P_{jtd}(K_{j-1}^k, j) = \lambda_j^{jtd} K_{j-1}^k \quad (25)$$

We do not know the value of λ_j^{jtd} in equation (25) at time j . Consequently, similar to what we have done with the Arrow-Debreu prices in equations (8) through (10), we express λ_j^{jtd} in terms of $j-1$. Therefore:

$$\lambda_j^{jtd} = e^{-r_{j-1}h} \left(\lambda_{j-1}^{jtd} + \sum_{i=0}^K \lambda_{j-1}^i (1 - z_{j-1}^i) \right) \quad (26)$$

The Arrow-Debreu price of a jump to default in period j , λ_j^{jtd} , is equal to the discounted value of the Arrow Debreu price of a jump to default in period $j-1$, λ_{j-1}^{jtd} , plus the marginal probability of default coming from the individual nodes in the layer $j-1$. Now that we know the value of λ_{j-1}^{jtd} in equation (25), we can easily solve the equation for the lower nodes.

We can use equations (24) through (26) and find the solution for the lower nodes in the tree and the derivation of the solution is much simpler, as shown in appendix C:

$$S_j^k = \frac{S_j^{k+1} (e^{r_{j-1}h} (P_{df}[K_{j-1}^k, j]) - \rho_{down}) - \lambda_{j-1}^k F_{j-1,j}^k (S_j^{k+1} - K_{j-1}^k)}{e^{r_{j-1}h} (P_{df}[K_{j-1}^k, j]) - \rho_{down} - \lambda_{j-1}^k z_{j-1}^k (S_j^{k+1} - K_{j-1}^k)} \quad (27)$$

with $\rho_{down} = \sum_{i=0}^{k-1} \lambda_{j-1}^i z_{j-1}^i \left(K_{j-1}^k - \frac{F_{j-1,j}^i}{z_{j-1}^i} \right)$ and $P_{df}[K_{j-1}^k, j] = P[K_{j-1}^k, j] - P_{jtd}[K_{j-1}^k, j]$

and $P_{jtd}[K_{j-1}^k, j] = e^{-r_{j-1}h} \left(\lambda_{j-1}^{jtd} + \sum_{i=0}^K \lambda_{j-1}^i (1 - z_{j-1}^i) \right) K_{j-1}^k$

Equation (27) reduces to the solution of BC (See Barle and Cakici, 1998, equation (15)) when we set z_{j-1}^k equal to one. Equation (27) above reveals the price of a default-free put option P_{df} . Carr

and Madan (2010) also use default-free stock processes in their (continuous-time) version of local volatility enhanced with a jump to default. Here the price of the default-free put option is equal to the price of a put option minus the price of the jump to default put option. There is an interesting relation between the price of a jump to default put option and the price of a CDS as indicated by Carr and Wu (2011)¹⁰, but we leave that empirical test to further research.

3.7 Requirements for the Implied Tree

As documented by BC, an implied tree should satisfy the following criteria: (1) correct reproduction of the volatility smile, (2) node transition probabilities (q) lying in the $[0,1]$ interval only, and (3) risk-neutral branching process (forward price of the underlying asset equals the conditional expected value of itself) at each step.

The tree is calibrated with option prices to fulfill requirement one. As documented by BC, implied trees sometimes violate requirement 2.

From the no-arbitrage condition in equation (2), we can deduce that q_j^i will be in the limit $[0,1]$ only when the ratio $\frac{F_{j,j+1}^i}{z_j^i}$ satisfies:

$$S_{j+1}^{i+1} > \frac{F_{j,j+1}^i}{z_j^i} > S_{j+1}^i \quad (28)$$

So that the stock price S_{j+1}^{i+1} is limited by the following inequality:

$$\frac{F_{j,j+1}^i}{z_j^i} < S_{j+1}^{i+1} < \frac{F_{j,j+1}^{i+1}}{z_j^{i+1}} \quad (29)$$

The stock price in period $j+1$ at node level $i+1$ should be in between the forward rate divided by the survivorship probability at j and node level i and the forward rate divided by the survivorship

¹⁰ The price of a digital default swap (a CDS that pays one dollar at default) is equal to the price of the jump to default put option divided by the strike price.

probability at j and node level $i+1$. If there is no default probability, then the survivor probability will always be equal to one and inequality (29) is equal to that derived by BC.¹¹

If the stock price does not satisfy inequality (29), we follow the override procedure suggested by Härdle and Mysickova (2008)¹², replacing it by:

$$\ln S_{j+1}^{i+1} = \ln \frac{S_j^i}{S_j^{i-1}} + \ln S_{j+1}^i \quad (30)$$

The logarithmic difference between the stock prices in this layer is equal to the logarithmic difference of the stock prices in the previous layer.

When the resulting stock price still does not satisfy (29), we apply the second override procedure suggested by Härdle and Mysickova (2008):¹³

$$S_{j+1}^{i+1} = \frac{1}{2} \left(\frac{F_{j,j+1}^{i+1}}{z_j^{i+1}} + \frac{F_{j,j+1}^i}{z_j^i} \right) \quad (31)$$

The stock price is equal to the average of the two forward rates.

3.8 Volatility Surface

Although in the Black-Scholes model the implied volatility is constant, in practice the implied volatility of an option depends on the strike price and the time to expiration. The implied volatility for at-the-money put options is usually lower than for (far) out-of-the-money put options. This effect is referred to as the “skew effect.” Also options with the same strike price, but different expiration dates trade at different implied volatilities. This is referred to as the term structure of volatility. Together the skew and term structure effects are described by the volatility surface. This volatility surface is a

¹¹ See equation (5) in their paper.

¹² Overrides occasionally happen and will have an effect on the implied volatility. In order to assess this effect we reverse engineer the implied volatility from the tree in Section 5.4 and show that limited overrides are needed when we define a skew.

¹³ For the DH model we also set a minimum value of b based on the equation (30). The stock goes either up to $e^{\sigma\sqrt{h}}$ or down to $e^{-\sigma\sqrt{h}}$. Then we can determine the minimum value for b so that equation (30) is satisfied. Setting this minimum value for b also for the implied models helps in finding the initial solution. For the other models, in a limited number of cases, it was also helpful to set a minimum value for b .

dynamic relationship: if markets are panicking, the skew might be steeper and when the company is reporting earnings on a short horizon, the term structure of volatility might be inverted.

The implied volatility surface is derived from quoted option prices that have different strikes and maturities. If we construct the implied tree, we might need strike prices or time to maturities that are not available directly from quoted option prices. Therefore we would like to have a continuous functional form for the implied volatility. Derman and Kani (1994, p. 10) provide an example of how to model the effect of the strike in the form of a deterministic function: the implied volatility increases linearly by 0.5 percentage points with every 10 point drop in the strike price. While several deterministic functions have been documented,¹⁴ we choose to follow Dumas et al. (1998) who run a polynomial regression on quoted option prices. The implied volatility σ is a function of the moneyness M and time to expiration T :

$$\sigma(M, T) = \beta_0 + \beta_1 M + \beta_2 M^2 + \beta_3 T + \beta_4 T^2 + \beta_5 MT \quad (32)$$

where MT is the cross-product of M and T and T is defined as the log of the time to expiration and we define moneyness M as:

$$M = \log\left(\frac{K}{F}\right) \quad (33)$$

where K is the strike price and F is the stock's forward price. Since T and M are both in log terms, the intercept can be interpreted as the implied volatility of a one year at-the-money option.

We estimate the regression given by (32) in Section 5.1 and then use the discrete volatility function when constructing the binomial tree.

3.9 The two other building blocks in the DH model

A CDS is an insurance contract, where the seller of the contract promises to pay out the credit loss on a bond when there is a credit event and the buyer of the contract promises to pay the periodic premium up to the time of any credit event (should one occur). The most common credit events as defined by the International Swaps and Derivatives Association (ISDA) are bankruptcy, failure to pay

¹⁴ See also Ait-Sahalia and Lo (1998) where they use a kernel regression.

and restructuring.¹⁵ DH use a standard CDS pricing model as described in, for example, Duffie (1999).

The two legs of a CDS are priced as follows:

Leg 1: The seller of protection promises to pay the recovery rate when a credit event occurs, so the expected present value, computed over all times (i,j) , for states i and times j , of this leg of the contract is:

$$B_N = \sum_{j=1}^N \sum_{i=1}^{M_j} Z_{j-1}^i (1 - z_j^i) D_j (1 - R_j^i) \quad (34)$$

where Z_{j-1}^i is the probability of surviving until state $(i,j-1)$, $(1 - z_j^i)$ is the probability of a credit event (a default for example) in state (i,j) , D_j is the discount function, and R_j^i is the recovery rate in state (i,j) .

Leg 2: For the buyer of the contract the expected present value of the premium payments over all states (i,j) is:

$$A_N = C_N h \sum_{j=1}^N \sum_{i=1}^{M_j} Z_{j-1}^i D_j \quad (35)$$

where C_N is the premium of the contract.

Then both legs should equal each other, $A_N = B_N$, allowing us to solve for the coupon C_N of the N -period CDS contract.

In the above model, we get the survival probabilities via the binominal tree, but we need to make an assumption for the recovery rate R_j^i . DH modelled the recovery rate as follows:¹⁶

$$R_j^i = \Phi(a_0 + a_1 \delta_j^i) \quad (36)$$

¹⁵ Next to these three credit events there are also Obligation Default, Obligation Acceleration and Repudiation/Moratorium (Markit, 2008, p. 27).

¹⁶ Das and Hanouna (2009, p. 1849) specify two alternative specifications for the recovery rate and conclude that the model is insensitive to the parametric form of the recovery function.

where $\Phi(\cdot)$ is the cumulative normal distribution, δ_j^i is the default probability, and a_0 and a_1 are parameters to be estimated. This functional form has the advantage that it is always positive (i.e., one cannot lose more than one owns). Then we can solve the model for a_0 , a_1 , and b given that all the other variables are market observables. We note that values ξ_j^i , δ_j^i , and R_j^i are all functions of the state variable S_j^i , and this induces a correlation between default and recovery rates. The sign of this correlation will depend on parameter a_1 .

4. Data

In this section we describe the data needed to estimate the model.

4.1 Choice of universe

To estimate the impact of the implied tree on the recovery rates of CDS spreads, we need data from companies that are listed on a stock exchange, have equity options, and have public debt outstanding so that there is an active CDS market. Additionally, we would like to have companies with a long track record and prefer to have a predefined list of stocks rather than select stocks for this study ourselves.

Consequently, we would like to have a universe of companies satisfying the above restrictions that is compiled by some external provider. However, we have found that indices are segmented in asset class as well as credit quality. For example, there is no fixed-income benchmark that includes only listed companies. Fixed-income benchmark constructors typically only include assets based on the fact that they have debt outstanding. The same is true for equity indices. For CDS indices, the main relevant criterion for a constructor such as Markit is trading volume. Additionally, we observe that many fixed-income benchmarks are segmented by either investment grade or high yield corporate issues.

We considered using all available CDS quotes from the Markit CDS database and filter on the listed companies; however, this information is not available from Markit. Moreover, we would also have many companies included only for a few months. While we considered using the Markit CDX index, this index also separates credit quality and not all constituents of the index have publicly traded

stocks. Another alternative that we considered was the NASDAQ 100 index, especially because the index is diverse in credit quality. Unfortunately, for this index not many stocks have CDS quotes. Therefore we choose the 30 companies that represented the Dow Jones Industrial Index (DJIA) as of September 24, 2012. We acknowledge that this universe is tilted towards the higher graded companies in the universe.

4.2 CDS data

We use the Markit Corporate Debt CDS database. This database contains the daily pricing history for single name CDS spreads from January 2001 to June 2014. We restrict our sample to the 30 stocks in the DJIA and take the corresponding spreads that (1) are denominated in U.S. dollars (USD), (2) are senior in tier, and (3) have a modified restructuring document clause.¹⁷ Although the database contains information on the full term structure of CDS spreads,¹⁸ we are only interested in tenors up to two years because listed stock options typically have a maximum time to expiration of two years. Three tenors are available for this part of the term structure: six months, one year, and two years.

Five-year tenors are seen to be the most liquid contracts; we compare in Figure 2 the availability of this contract with the three tenors that we want to use. At the start of the Markit database in January 2001 not all 30 companies are represented or have a full CDS term structure. The Markit CDS spread is a composite of quotes from individual contributors (brokers) and so if there is no CDS spread available it would not directly mean that there are no contributors, but rather that the Markit composite rules are not fulfilled.¹⁹ At the end of January 2001, only 13 stocks had a CDS spread for the five-year tenor, but none of the 30 stocks had CDS pricing for the six-month tenor. In subsequent years more and more contracts were quoted in the market and appearing in CDS database. At the end of July 2003 all 30 stocks had a CDS quote for the five-year tenor. It took however until the end of August 2005 before the CDS spreads of all 30 stocks were consistently available in the

¹⁷ This document clause is the most prevalent for US Investment Grades (Markit, 2008, p. 28) until 2009. After the CDS Big Bang the no restructuring clause became prevalent (Markit, 2009). For the full period 2005 till 2014 the modified restructuring is the most quoted contract (see Exhibit 1 in Jansen and Fabozzi, 2017).

¹⁸ The database contains tenors (in years) of 0.5, 1, 2, 3, 4, 5, 7, 10, 15, 20 and 30. Not all those tenors are always filled.

¹⁹ The composite price is the average of the prices provided to Markit by its contributors, once those prices failing any one of the data quality tests have been excluded. In order to form a composite, Markit requires at least three distinct contributors submitting curves of which at least two pass all data cleaning tests Markit (2011).

database.²⁰ The one- and two-year tenors followed the five-year tenor closely in availability, with only the six-month tenor lagging. When all five-year tenors had CDS spreads available at the end of July 2003, only 12 six-month tenors had a CDS price. The availability of the five-year tenors stabilized after August 2005; however, the availability of six-month tenor quotes varied widely after this date. In August 2005 there were 26 CDS spreads, but only 20 in January 2007.

<<Insert Figure 2 about here>>

From the fact that six-month tenors are less available for the period to be analyzed we can surmise that this tenor is less liquid. Markit Corporation also provides a quantitative measure for the quality of the tenors in the form of a 7-grade letter combination (Markit, 2017). The highest rating of AAA is given a score of zero and indicates that the data quality scores highest in the four categories.²¹ If the data-quality does not fulfill these highest requirements, points are added (de-merited). For example if there are between five and eight contributors to a tenor (instead of more than nine), one point is added and the tenor will get a data-quality rating of AA. To achieve a rating, the CDS tenor composite must have at least three clean contributions (otherwise it is not rated, i.e. assigned a rating of NR). Markit (2017) states that particular confidence can be placed in ratings of BBB or higher, for which a very minimum of three contributions are required alongside the highest score in the other three categories.

Table 1 shows the average data-quality rating for the CDS spreads in our universe for the six-month, one-year, two-year, and five-year tenor.²² Each month-end we transform the letter score into a numeric score following the scaling of Markit Corporation²³ and then we aggregate over the period before transforming it back into the letter grade score. Three things are quite apparent if we look at the average ratings for the full period in columns one to four: (1) if we take the mean over all companies in the last row (columns one to four), the one-year and two-year tenors with a rating of A are as liquid

²⁰ There is one exception in that MSFT did not have a price for the five-year tenor at the end of February 2009 (seen as the height of the credit crisis).

²¹ These four categories are (1) Quantity of clean contributions, (2) Tightness of the tradable range, (3) Freshness of data, and (4) Ease of modelling.

²² The data-quality ratings start in October 2005, while we report here from August, so for the first two months we don't have a rating and thus the average over October 2005 till June 2014 is used.

²³ Markit uses the following transformation table between letter and numeric (between brackets): AAA (0), AA (1), A (2), BBB (3), BB (4), B (5), CCC (6-9) and NR (11-12). We use the mean for CCC (7.5) and NR (11.0).

as the five-year tenor, (2) the six-month tenor (in column one) is clearly the least liquid with a BB rating which indicates a wider range of spreads from (composite) contributors, stale prices or no full curve, and (3) data quality varies even for five-year tenors (in column 4) from AA to CCC. The main reason why Microsoft (MSFT) was the most illiquid over the whole curve is that it had no or very little debt outstanding.²⁴

To determine if the data-quality varies over time, we have separated the sample into three subperiods of equal length. The first subperiod (August 2005 to July 2008), in columns five to eight is characterized by very liquid five year-tenors, liquid one and two-year tenors, and illiquid six-month tenors. In the second subperiod (August 2008 to June 2011), the period of the credit crisis, in columns nine to 12, liquidity dried up dramatically for all tenors. The average for the six-month tenor in column nine decreased to CCC, indicating a wider range of spreads from (composite) contributors, stale prices, and no full curve. However, all tenors still had a quote, indicating that there were at least three contributions. The last subperiod (July 2011 to June 2014) in columns 13 to 16 shows that markets are liquid again. The six-month tenor even has the highest liquidity for all subperiods and is classified as BBB, a rating class Markit (2017) asserts is a rating class of particular (high) confidence. This is because a minimum of three clean contributions are required alongside the highest score on the qualitative tests. Five-year tenors, in column 16, are liquid as well although not as liquid as in the first subperiod (column nine).

Lower liquidity may result in less accurate (composite) prices, biasing the estimates of our model. Because we use composite prices that are comprised of at least three contributors instead of broker prices, estimates will be less biased. Also, since our model is parsimonious, less liquid data will only impact that particular estimate and not a series of estimates as in a time-series. Additionally, we have divided the data sample into three subperiods to monitor the effects of liquidity on the estimated parameters. Moreover, we will primarily look at two-year recovery rates and default probabilities, which will be less affected by biases in the six-month tenors because in addition to that tenor, we also use the liquid tenors (i.e., 1 and 2 years) for the estimation.

²⁴ Bloomberg reports that Microsoft has no outstanding debt between 2001 and 2009.

We decided to start the sample in August 2005, the date from when the five-year tenor is consistently available. When a tenor is not available, we use the average of the longer and shorter tenor and when no shorter tenor is available, we use the higher tenor. We also incorporate the 18-month tenor as done in DH by interpolating the quoted prices. For each company we therefore have four points on the CDS curve.

4.3 Equities, options, and yield curves

We collect all other data that are needed to estimate the model from the Ivy DB Option Metrics²⁵ database that is available on Wharton Research Database. The data in OptionMetrics is available for a much longer history²⁶ than is available for the CDS data and therefore we match it to the length of the CDS data: August, 31 2005 till June 30, 2014. We obtained four different files for all 30 stocks in our universe: (1) the volatility surface, (2) standardized options, (3) security prices, and (4) zero-coupon yield curves.

From the volatility surface file we take the implied volatility, implied strike, and the time to expiration at each month end for all 30 companies for four different times to expiration.²⁷ In the standardized file we select the forward price (that includes the dividend and the interest rate) at each month end for all 30 companies for the four different times to expiration. Sometimes the standardized file does not contain all times to expiration. Then we compute the forward price ourselves by first implying the dividend from the last longest available forward rate:

$$d_{0,t} = \log\left(\frac{F_{0,t}}{f_{0,t}}\right)/t \quad (37)$$

where $d_{0,t}$ is the annualized dividend between between time $t=0$ and time t (in years), $F_{0,t}$ is the forward price and $f_{0,t}$ is the interest rate between time $t=0$ and t .

Then we compute the missing forward rate using the annual dividend rate from expression (37):

²⁵ We downloaded for each of the 30 companies all the information in the menu (1) standardized options, (2) volatility surface, and (3) securities prices.

²⁶ Data are available from January, 1 1996 to August 31, 2014.

²⁷ The four times to expiration are six months, one year, 18 months, and two years.

$$F_{0,t+0.5} = S_0 e^{(f_{0,t+0.5} - d_{0,t})(t+0.5)} \quad (38)$$

where we increase time t by 0.5 (six months).

We only include options that are out-of-the-money since in-the-money options trade less and are more expensive. Therefore, we only include call options where the moneyness is greater than zero ($M > 0$) and put options where the moneyness is smaller than zero ($M < 0$).

The zero-coupon yield curve is only available in certain standard maturities.²⁸ We simply interpolate the zero-coupon curve to match the maturities with the tenors of the CDS.

5. Empirical Results

In this section we report the results of the tests of our model.

5.1 Discrete Volatility Function

We run the cross-sectional regression in equation (32) for each of the 30 companies at each month end for the period August 2005 to June 2014. We only included the implied volatilities with a time to expiration of 6, 12, 18, and 24 months to match the tenors from the CDS available quotes. In Table 2, we report that the regressions explain most of the variation as can be seen in column 14. All R-squares exceed 0.9 (the average is 0.97). In the preceding columns, we report the average of the coefficients over the 107 regressions. Column one (C) shows the constant in equation (32) and represents the volatility of an at-the-money option with a one-year time to expiration. For example, Alcoa (AA) has an average volatility of 38.7% over this period. This volatility is not constant, as the standard deviation of the monthly coefficients is also large (13.9%).

Column three (M) shows the results for the (log) moneyness. All stocks have a negative sign for the moneyness, which clearly confirms that there is a volatility skew. The average of the moneyness coefficients varies from -10 to -22. For IBM this indicates that the implied volatility of a put option that has a strike price that is approximately 10% below the forward price, the implied volatility is 1.94% higher.

²⁸ Maturities are 9, 15, 50, 78, 169, 260, 351, 442, 559, 624, 715, and 806 days.

Column five (M^2) shows the results for the squared moneyness variables. This variable represents the smile effect of the implied volatility. All average coefficients are positive and range from 6 to 22.6 and we see that the standard deviations are sometimes high. If we take the same example as above, the effect of this variable on the implied volatility for AA is only an increase of 0.14%.

Column eight (T) shows the coefficients from the time to expiration. All but one (i.e. AA) coefficients are positive and are around one. Therefore, if we extend the time to expiration from one year to two years, we increase the implied volatility of IBM on average by 0.49%. When we shorten the time to maturity, we decrease the implied volatility.

Column ten (T^2) shows the coefficient from the squared time to expiration. The results are more mixed, most are negative but there are also some positive average coefficients. The effect from extending the time to expiration from one year to two years on the implied volatility when we have a coefficients of -0.5, as for example for IBM, will be a decrease in implied volatility of -0.24%. If we sum the two effects for the time to expiration for IBM, the impact of an extension of the time to expiration from one year to two years will be +0.25%.

Column twelve (MT) shows the interaction effect. All average coefficients are positive, ranging from 3.4 to 9.6. This implies that the skew declines as maturity increases.

<Insert Table 2 about here>

5.2 Fitting the model

We can now calibrate the three parameters of the model (a_0 , a_1 , and b) using a least squares fit of the model CDS spreads (CDS_k^{Model}) to the market CDS spreads (CDS_k^{Market}), where k indicates the tenor number. The function that we minimize over is the mean squared error (MSE), but we report here the relative root mean square error (RRMSE) as this is the same measure used in DH:

$$RRMSE = \frac{\sqrt{\frac{1}{4} \sum_{k=1}^4 (CDS_k^{Model} - CDS_k^{Market})^2}}{\frac{1}{4} \sum_{k=1}^4 CDS_k^{Market}} \quad (39)$$

We fit the DH model and then the implied model with a jump to default where the local volatility is defined by the deterministic volatility function in equation (32). We define four different specifications for the implied volatility function²⁹ in equation (32):

- (1) The constant volatility (CV) model where we include only the β_0 coefficient and set all other β coefficients to zero. This model differs from the DH model in that it has constant volatility at each node, where the local volatility in the DH model increases as the stock price becomes lower.
- (2) The term structure (TS) model where we include β_0 , β_3 , and β_4 coefficients and set the β_1 , β_2 , and β_5 coefficients to zero. Here the local volatility is constant in a certain layer in the tree, but differs between layers.
- (3) The skew (SK) model where we include the β_0 , β_1 , and β_2 coefficients and set the β_3 , β_4 , and β_5 to zero.
- (4) The volatility surface (VS) model that includes all β 's.

We calibrate the model parameters for each month for the period August 2005 to June 2014. The averages over those months for the four specifications above are reported in Table 3. For example for Boeing Company (BA), the average RRMSE for the full period in the DH model is 1.7%, that is, the standard deviation of the error is 1.7% around the mean of the four CDS spreads (six months, one year, 18 months and one year). In the last row we report the average RRMSE over the individual companies which is equal to 2.0% for the DH model (bottom row, column one). DH (2009, p. 1845) report a RRMSE of 4.05% for the companies with the lowest default probability over the period January 2000 to July 2002. As indicated earlier, we also separated the sample into three subperiods of equal length. Although, the subperiods have different levels of credit spreads and different liquidity,

²⁹ In theory we could run a separate regression for each model, but we choose for simplicity to use the regression with all variables included and then select the β 's from that model for the sub-model.

we see that the average RRMSE over all companies for the DH model decreases in time from 2.3% in the first subperiod to 1.6% in the last subperiod.

<Insert Table 3 about here>

The first specification that we consider, the CV model (in column two), has a RRMSE that is slightly above that of the DH model for the full period (2.2% versus 2.0%). Also we see that the RRMSE decreases with time as the error in the last subperiod is lower than in the first subperiod.

The TS model (in column three) has the lowest RRMSE of all models. The average RRMSE is slightly better than the DH model both for the full sample as well as all subperiods.³⁰ The TS model has a flat volatility structure, but the volatilities differ by tenor. On an individual company level, the TS specification has for the majority of the companies a lower RRMSE than the estimate with the DH model. Also for the TS specification, the errors seem to decrease in time as the average RRMSE for the first subperiod is 2.5%, the second subperiod is 1.9%, and the last subperiod is 1.4%.

The mean RRMSE for the SK and VS specifications where the skew parameters (β_3 , β_4 , and β_5) are included are all well above the DH model, while the errors do not have a specific trend in time. The error for the VS model is mostly stable at 2.3% both for the full period (column four) as for all subperiods (columns 9, 14 and 19). The error for the SK specification is a bit lower than for the VS specification, but for the subperiods there is no clear pattern.

Based solely on the perspective of the RRMSE metric, it seems that the CDS spreads are priced only in the dimension of time and not in the dimension of moneyness (i.e., skew). The fact that the RRMSE increases if we add the volatility skew suggests that the CDS market prices a different skew. Potentially we can imply the “CDS implied volatility” as a counterpart of the “option implied volatility” with our model. Some efforts, but from a different angle, have been made already by Guo (2016), Kelly, Manzo, and Palhares (2017), Culp, Nozawa, and Veronesi (2017). The challenge we have with our model is that we have only four tenors on the CDS curve and therefore have an over-

³⁰ Although the table shows for formatting reasons an average MSE of 2% for both models, the average MSE for the TS model is 1.98% in the subsample August 2008- June 2011, while average MSE for the DH model is 2.01%.

identification problem if we fit the eight parameters (a_0 , a_1 , b , and the five β 's) of the joint model. Consequently, we need more points on the CDS curve, for example, a curve that is specified per quarter or even per month instead of half a year. We leave this exercise for further research. Another interesting area for further research is to see if the mispricing with the skew is exploitable; that is, can we construct a trading strategy that buys the underpriced options/CDS and sell the overpriced options/CDS? Carr and Wu (2011, p. 488) mention that market participants see the arbitrage strategy of selling credit insurance through CDS and hedging the exposure by buying (deep) out of the money put options on the same company as a profitable strategy.

5.3 Estimated parameters

Although the differences between the RRMSE of the different models are close to each other, the estimated parameters are different. In Table 4, we report the average values of the different estimated parameters for the full sample and also the average of the stock price for the full period. If we compare the estimated values for the DH model with the results in the DH paper (p.1845) we see that the value of our parameter b (1.24) at the bottom of column three is close to the one in their paper (1.34) but that the values for a_0 (3.9 versus 2.1) and especially a_1 (-454 versus -4.1) are very different. The large difference for the a_1 coefficients can be attributed to the differences in model structures.³¹

<Insert Table 4 about here>

Also as we know from DH (p. 1844), parameter a_0 has no economically determined sign in the model, but the other two parameters do. Parameter a_1 is negative in the full sample for all companies and model specifications and from equation (36) we know that this results in an inverse relation between the default probability and the recovery rate: if the default probability goes up, the

³¹ There are three main differences in model structure between our model and DH that causes these large differences in the a_1 coefficient. First, our model includes half year tenors whereas DH uses full year tenors. As a result, the default probabilities are smaller and therefore the (absolute) a_1 coefficients should be higher to arrive at the same recovery rate. Second, our model only includes tenors up to 2 years compared to up to 5 years in the DH model. As a result, the lowermost node in the last layer of the binomial tree has a higher price and therefore a lower default probability and therefore the model can set a higher absolute value for the (absolute) a_1 parameter. Third, our model includes dividends and therefore the forward prices in the tree are lower.

recovery rate goes down. Parameter b is positive (as required) for all companies and models and we know from equations (3) and (4) that as stock prices fall, the probability of default rises.

If we compare the b parameters in the DH model (column three) with the ones from the implied models we see that they are lower with respect to the constant volatility models (column six and nine) but more or less the same with respect to the specifications that include the skew (columns 12 and 15). What we have seen in Section 2 and also in Figure 1 is that a decrease in the value of the b parameter will increase the ATM volatility and create a skew. Therefore, allowing a skew (either in the DH models or the SK and VS) will reduce the value of the parameter b and therefore by equations (3) and (4) increase the default probability (i.e., lowers the survival probability). However, although the DH, SK, and VS models have the same value for b , it does not mean that the default probabilities are equal, as the underlying binomial tree can be completely different. We will look at this in the Section 5.5.

The value of b is more or less the same between the DH model and the SK and VS specifications, but the a_0 and a_1 parameters are different. The fact that these parameters are different gives us some indication that the recovery rates are different; however this is also dependent on the underlying binomial tree as we will see in Section 5.5.

5.4 Reverse engineering the implied volatilities from the tree

Although we have set the DH model as the benchmark for the different specifications of our implied model, we have to say that this is not a truly fair benchmark. It is not fair because the DH model is less restrictive in volatility. As we have shown in Section 2, a lower value of b in the DH model will create a volatility skew, but also increase the at-the-money implied volatility of the model to potentially unrealistic high levels. The implied models are by construction mimicking the implied volatility of the option market. So even if the DH model produces a lower error than an implied model, the implied model is more realistic in volatility terms.

<Insert Figure 3 about here>

Given that we have estimated the different models by calibrating the three parameters, we can also inspect the produced implied trees and reengineer the implied volatility from each tree to check if the developed model works. Therefore, although we could look at an individual node level and calculate the local volatility of that node, it is complicated to calculate the local volatility over several nodes. Instead, we choose to calculate the binomial option price with Arrow-Debreu prices for the call option via equation (5) and for the put option via equation (24) and then calculate the implied volatilities using the Black-Scholes formula.

We also know that the binomial model is only an approximation for the Black-Scholes formula when the time step goes to zero. In our case, we have a time step of 0.5, which introduces an error in the binomial option price, and thus ultimately in the implied volatility that is calculated via the Black-Scholes formula. However, as suggested by BC, we used Black-Scholes option prices to estimate our models, so for specific exercise prices in the tree, there should be no error. Therefore, if we want to look at the implied volatility for the two-year options, we can use the four exercise prices that we have used to estimate the model and compare those with the discrete volatility function. For other exercise prices, we will have an error in the option prices and therefore an error in the implied volatility. Ideally, we would choose the exercise price ourselves, since we would like to compare how well the implied model is in pricing the volatility tree over time. Since we cannot do that without introducing a large error, we choose the exercise prices that we have used to optimize the model and then interpolate these exercise prices to set levels of moneyness (-0.3,-0.15, 0.15 and 0.3). We have chosen this small band of moneyness to ensure that most trees fit. In such cases where the moneyness does not fit, we set the value of the implied volatility equal to the value from the discrete volatility tree.

We compute the implied volatilities for all months in our database for each company and then first average for each company and then average over all 30 companies in the DJIA. In Figure 3 we plot the results for the DH, TS, and VS models. Panel a shows the result for the full period for the TS model and as we know, this model has a flat volatility. First we plot the line that represents the Discrete Volatility Function (DVF) and then we mark the points that represent the re-engineered

implied volatility for the TS model for the four exercise prices. The four implied volatilities from the TS model correctly reproduce the volatility as they are all on the DVF. The DH model is as we have seen earlier in Figure 1 is well above this curve for the ATM options and is also skewed. From analyses we have done for the three subperiods³², the TS specification correctly represents the implied volatilities for the positive moneyness options and it is marginally off for the negative moneyness options. It seems that it is more difficult to keep constant volatility in the lower part of the tree than in the upper part of the tree. That can, for example, also be seen if we go to the VS specification in panel b, where the DVF has a skew and the VS model is able to correctly reproduce the volatility both with positive moneyness as well as negative moneyness. The DH model also lies well above the implied model indicating that the DH model has a higher volatility than the option market.

What is clear from the above analysis is that the implied model with a jump to default correctly reproduces the DVF in general and if we specify a DVF with a skew the algorithm doesn't need many overrides as the implied volatility lies exactly on the DVF.

5.5 Default probabilities

The default probability at each node in the tree can be easily calculated by using equation (4). To calculate the default probability for over one layer of nodes we need to do backward recursion with the probabilities. Das and Hanouna (2009, equation (15)) give the forward probability of default as:

$$\delta_{j,j+1}^{forward} = \sum_{i=0}^j p_j^i \delta_j^i \quad (40)$$

The forward default probability is given by the default probabilities (δ_j^i) discounted by the cumulative probability of reaching that state (p_j^i). The forward probability gives the probability that the company defaults between time j and $j+1$.

We can also write the cumulative probability:

³² The respective graphs are available upon request from the authors.

$$\delta_{j+1}^{cum} = \sum_{i=0}^j p_j^i \delta_j^i + \sum_{i=0}^j (1 - p_j^i) \quad (41)$$

The cumulative probability is equal to the forward probability and the probability that the company already defaulted in an earlier period (layer).

Now we can use equation (41) to calculate the two-year default probability for the companies in the DJIA for the different model specifications and subperiods. Column one in Table 5 shows the results for the DH model for the full period. Note that in contrast to the convention in the interest rate and spread markets, these are not annualized probabilities but cumulative probabilities. The average cumulative default probability for the DH model for the full period is 4.1% (bottom line of column one), thus approximately 2% per year. Two companies have rather high default probabilities – Alcoa (AA) and American Express (AXP) – while the rest of the companies are substantially lower. We see also that the default probability in the first subperiod (column six bottom line) was the lowest of all subperiods, while the second subperiod (that included the credit crisis) has the highest default probability (column 11 bottom line).

<Insert Table 5 about here>

Focusing on the different specifications for the implied models, we see that the flat volatility models (CV and TS) have substantially lower default volatilities than the ones that include the skew. We also see that the specification that includes the full volatility surface (VS, bottom row of column 5, 10, 15 and 20) has the highest default probability of all specifications. The effect of the skew is thus that the default probability increases (at the expense of the recovery rate as we will see later). On an individual basis, the VS specification is almost always higher than the TS specification, except in the first subperiod. This suggests that the skew was much less apparent in the first subperiod than in the other periods as we have seen in Figure 3.

5.6 Recovery rates

Das and Hanouna (2009, page 1853 equation (14)) state that the forward recovery rate can be expressed as:

$$R_{j,j+1}^{forw} = \sum_{i=0}^j p_j^i R_j^i \quad (42)$$

The forward recovery rate is equal to the recovery rate discounted by the probability of reaching that state.

We have a problem with this equation as the probabilities p_j^i in equation (42) do not sum to one. The reason is they only sum over the states when the company has not defaulted yet. Therefore we propose an improvement of the equation in Das and Hanouna (2009):

$$R_{j,j+1}^{forw} = \frac{\sum_{i=0}^j p_j^i R_j^i}{\sum_{i=0}^j p_j^i} \quad (43)$$

where we divided the equation (43) by the sum of the probabilities: $\sum_{i=0}^j p_j^i$.

The problem in equation (42) can be easily seen in Figure 4 where we simulated two simple cases with no discounting (interest rates are zero). In the first case, we assume that the recovery rate is always 100% when a_0 equals 100, a_1 equals zero, and b is set to one so there is some default probability. When we run equation (42) in this case, the term structure of forward rates is downward sloping, purely because there is some default probability. When we adjust with equation (43), we see that the term structure is stable at 100%, which should be the correct forward recovery rate.

<Insert Figure 4 about here>

In the second case we let the term structure of recovery rates vary by setting a_0 to 3 and a_1 to -200 . In this simulation, the term structure of recovery rates is downward sloping for both equations but for different reasons. For equation (42) it is downward sloping as a result of the default probability and $a_1 < 0$, while for equation (43) it is only as a result of $a_1 < 0$. The a_1 parameter thus determines the slope of the recovery rate term structure.

In Figure 5 we have applied equation (43) to the estimated DH model and some specifications of the implied model (TS and VS). We can use equation (43) to calculate the marginal (or forward) recovery rates. We start with the forward rate at the initial node for time period 0.5. We calculate

equation (43) for all dates for each company and then average over the months and companies. Then we continue for the 0.5-year forward rate at point 0.5 and repeat the same procedure for all months and all companies. The results in Figure 5 show that the implied model with the VS specification has the highest recovery rate while the implied model with the term structure specifications has the lowest term structure of recovery. The 0.5-year forward rate at 1.5 year in panel a of Figure 5 is 72.1% for the VS specification and 64.9% for the TS specification, while the DH model shows a recovery rate of 69.9%. From panels b, c, and d, we also see that these recovery rates change over time and for the different models. The recovery rate of the DH model is the most volatile. In all panels we see that the forward curve is downward sloping, which is something that we expected given that the average a_t parameter is less than zero.

The term structure of the recovery rates for all companies is downward sloping for all specifications. This result supports what Das and Hanouna (2009) found. Doshi, Elkamhi, and Ornthalalai (2018) report a downward sloping recovery rates as well except during the financial crisis where they found an inversion point at around 24 months. This suggests a higher recovery rate for longer tenors given short-term survival. However, similar to our findings the first 24 months in their study were also downward sloping.

Instead of calculating the forward recovery rate we can also calculate the j -period (cumulative) recovery rate, that is defined as the implied recovery rate for a firm between time is null and time is j :

$$R_{0,j}^{cum} = \sum_{i=0}^j p_j^i R_j^i + \sum_{i=0}^j (1 - p_j^i) R_{j-1}^{cum} \quad (44)$$

The cumulative recovery rate is equal to the recovery rates of the nodes that have not defaulted yet plus the cumulative recovery rate in the last period weighted by the (cumulative) default probability.

<Insert Table 6 about here>

The effect on the recovery rates can be seen in Table 6, where we depict the two-year recovery rate calculated with equation (44). For the full period, on average the two-year recovery rate in the DH model is 75.7% (bottom row column one), the constant volatility specifications (CV and TS) are a

bit lower at 70.2% and 70.8%, respectively. DH report a comparable average recovery rate of 73.55%. We have in our sample somewhat better quality companies³³ and analyze a different period. In the subperiod that is closest to the period in DH study, we report a recovery rate of 78.8% (bottom row column six). Another reason for this difference is that we only include tenors up to two years while the DH study includes tenors up to five years. With a downward sloping recovery rate curve this results in a higher two-year recovery rate. As we might have expected from the idea that in bad times recovery rates are lower because companies have to sell the assets at fire sale prices, we see that in the second subperiod the recovery rates are much lower than in the other subperiods. The recovery rate for the DH model drops from 78.8% in the first subperiod to 68.2% in the second subperiod (bottom row column 11) and for the CV specification the drop is even more dramatic, from 75.2% (bottom row column seven) in the first subperiod to 61.0% (bottom row column 12) in the second subperiod. What can also be observed from Table 6 is that the recovery rates for models that include the skew are more stable. The recovery rate drops only from 77.3% (bottom row column 10) in the first subperiod to 75.6% (bottom row column 15) in the second subperiod, while the last subperiod is even higher at 81.4% (bottom row column 20).

It is clear from the values shown in Table 6 that the recovery rates are higher than the usual market practitioners' recovery rate assumption of 40%. Bakshi, Madan, and Zhang (2006) show that risk-neutral LGDs are lower than their physical counterpart so therefore these results are confirming this observation. Le (2015) shows that the average (5-year) recovery rate for companies with the same credit quality as in our sample (S&P A+ rating) in a CEV model with jump to default is 85% for the period September 2001 to June 2005. Elkamhi, Jacobs, and Pan (2014) estimate an average recovery rate of 53.79% for the period October 2004 to June 2009 for their sample of 152 firms.³⁴ If we only select the 14 firms in their sample that were also part of the Dow Jones Index in our sample, we can deduce from Table 3 in their paper that the average recovery rate is 67.41%.³⁵ We found in the first

³³ The average S&P rating of the companies in our sample is A+.

³⁴ Firms were selected on the basis of their inclusion in the CDX index.

³⁵ The fact that the recovery rate increases from 53.79% (for the full sample) to 67.41% (for the 14 firms) agrees with their finding that the recovery rate is positively correlated with firm size, as companies in the Dow Index have a higher market capitalization.

subperiod (2005-2008) for the same 14 firms a recovery rate of 77.65%³⁶, while in the second subperiod (2008-2011) a recovery rate of 67.53%. Although on average the recovery rates are similar, the cross-sectional variation is much higher in Elkamhi, Jacobs, and Pan (2014). We estimate an average standard deviation of 6.67% in the second subperiod, while Elkamhi, Jacobs, and Pan (2014) estimate a standard deviation of 18.48%. Our results are in line with Das and Hanouna (2009) and Le (2015) who report a cross-sectional variation of 6.5% for the period 2002-2005.

We can also inspect the time-series behavior of the average default probabilities and recovery rates. For each month end, we take the average for all firms of the two-year default probabilities and two-year recovery rates and we plot them in Figure 6. In panel a we see that the average recovery rates for the 30 companies vary from 48% to 86%. When we compare the DH model with the VS specification, we see that the VS model is more stable: for example it varies from 60% to 86% while the DH model varies from 51% to 86%. In panel b we see that the recovery rate is initially stable and then from May 2007 it declined, while default probabilities were unchanged. They continued to go down for the TS and DH models until October 2008 and then bounced after they experienced their lowest point in January 2010. The VS specification had its lowest value much earlier, in July 2008, and was much more stable in the period thereafter. This stability of the VS specification is clearest in panel d, where the recovery rate stays almost all of the last subperiod at around 80%.

The default probabilities are also not equal in the different models. In general, the VS specification shows the highest default probability.

Including the full information from the option market with the VS model increases both the default probability and recovery rate in comparison to the DH model.

<Insert Figure 6 about here>

³⁶ The average is taken over the four different specifications.

6. Conclusion

The CDS spread is the product of the PD and the LGD. Since only the CDS spreads are known from market prices, the LGD and PD are not separately identified unless some structure is imposed. Das and Hanouna (2009) developed a model (“the hybrid tree”) that identifies PD and LGD jointly by combining information from the CDS and equity markets. Their model is easy to calibrate and shows good performance. One deficiency of their model is that it fails to include richer information from the options market. The model only includes the ATM volatility while the implied volatility in the option market shows a skewed pattern in relation to the moneyness and a term structure pattern in relation to maturity. Also, as we show in this paper, their model produces, even for ATM options, unrealistic high volatility levels when compared to option implied volatility.

To overcome the deficiency in the Das-Hanouna model, we develop a new type of binomial tree, an implied tree with a jump to default, that exactly prices the option volatility surface. Our framework allows us to estimate five different model specifications over different volatility specifications, while nesting the Barle and Cakici (1998) model.

In the empirical part of the paper we find that our model mimics the different specifications of the implied volatility curve well. Surprisingly, the specification with the lowest error was not the specification that includes the full volatility specification (e.g., both the term structure and skew), but the specification that includes only the term structure of volatilities. One explanation for this could be that the option and CDS markets insure different elements: a CDS is triggered when there is a credit event while the payoff of an option is dependent on the stock price. Another explanation can be that one of the two markets is more efficient (includes more information) and therefore is leading the other market. Therefore, an interesting area for future research is to compute a CDS implied volatility surface with the model we developed. The dynamics of the CDS implied volatility and option implied volatility surfaces could help us in answering questions about efficiency and integration.

Our main theoretical contribution to the literature is that we have found an elegant extension to the implied binomial tree class of models by adding a jump to default. In addition to that theoretical

contribution, market participants can use the model to gain more insight into the decomposition of the credit spread into LGD and PD. Structurers who focus purely on recovery rates alone can gain important insights since for example the model with term structure specification produces lower LGDs than the Das and Hanouna (2009) model as well as a LGD that is based on more realistic levels of volatility. Additionally, structurers (or traders) can use the model to price longer term options. The CDS market is much richer than the options market because there are actively traded vehicles up to 30 years in tenor while time to maturities in the option market typically only goes out to two years. We can use these longer tenors in our model to price longer-term options. Finally, regulators and bank risk managers can use the model for more accurate parameters in complying with Basel 3 and IFRS9.

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Table 1

Data Ratings for Credit Default Swaps

The Markit Corporation provides a quantitative measure for the quality of the data in the form of a 7-grade letter combination. The highest rating AAA is given the score zero and indicates that there are nine or more contributors for the composite, a tight tradable range, actual data and a full curve. If the data quality is less in one of these four categories, points will be added to the score. To achieve a rating, the CDS tenor composite must have at least three clean contributions (otherwise not rating, i.e. NR). Particular confidence can be placed in ratings of BBB or higher, for which a very minimum of three contributions are required alongside the highest scores on the other three categories. For each month-end we transform the letter score into a numeric score following the numeric scaling of Markit Corporation. Then we average over the months and firms and transform the numeric scaling back into the letter combination.

Ticker	Full period				August 2005 - July 2008				August 2008 - June 2011				July 2011 - June 2014			
	(1) 6M	(2) 1Y	(3) 2Y	(4) 5Y	(5) 6M	(6) 1Y	(7) 2Y	(8) 5Y	(9) 6M	(10) 1Y	(11) 2Y	(12) 5Y	(13) 6M	(14) 1Y	(15) 2Y	(16) 5Y
AA	BB	A	A	AA	A	AA	AA	AAA	CCC	BBB	A	A	BBB	A	A	A
AXP	BBB	A	AA	AA	A	AA	AA	AAA	BB	A	A	AA	A	A	A	AA
BA	BB	A	A	AA	A	AA	AA	AA	B	BBB	A	A	BBB	A	A	AA
BAC	BBB	A	A	A	A	AA	AA	AA	BB	A	A	A	BBB	A	A	A
CAT	BBB	A	A	AA	A	AA	AA	AA	BB	A	A	AA	BBB	A	A	AA
CSCO	CCC	BBB	BBB	A	NR	BBB	BB	BBB	CCC	BB	BBB	A	BBB	A	A	AA
CVX	BB	A	A	A	BB	A	A	AA	CCC	BBB	BBB	A	BBB	A	A	A
DD	BB	A	A	AA	BBB	AA	AA	AA	CCC	BBB	BBB	A	A	A	A	AA
DIS	BB	A	A	AA	BBB	AA	AA	AA	B	A	A	A	A	A	A	AA
GE	B	BBB	BBB	BBB	B	BBB	BBB	A	CCC	BB	BB	BB	BB	BBB	BBB	BBB
HD	BBB	A	A	AA	BBB	AA	AA	AA	BB	A	A	A	BBB	A	A	AA
HPQ	BB	A	A	AA	BBB	AA	A	AA	CCC	BBB	A	A	BBB	A	A	AA
IBM	BBB	A	A	AA	BBB	AA	A	AA	BB	BBB	A	A	BBB	A	A	AA
INTC	CCC	B	B	BB	B	BBB	BBB	A	NR	CCC	CCC	CCC	CCC	A	A	A
JNJ	BB	BBB	A	A	BB	A	A	AA	CCC	BBB	A	A	BBB	BBB	A	A
JPM	BBB	A	A	AA	BBB	AA	AA	AA	B	A	A	A	A	A	A	AA
KO	B	BBB	BBB	A	B	A	A	A	CCC	BB	BB	BBB	BBB	BBB	BBB	A
MCD	BB	A	A	A	BB	AA	AA	AA	CCC	BBB	BBB	A	A	A	A	A
MMM	B	A	A	A	BB	A	A	AA	CCC	BBB	BBB	A	BBB	BBB	BBB	A
MRK	BB	A	A	A	BBB	AA	A	AA	CCC	BBB	A	A	BBB	BBB	A	A
MSFT	CCC	CCC	CCC	CCC	NR	CCC	CCC	B	CCC	CCC	CCC	CCC	CCC	CCC	CCC	B
PFE	BB	A	A	AA	BBB	A	AA	AA	B	BBB	A	A	BBB	A	A	AA
PG	B	BBB	BBB	BBB	BB	A	A	AA	CCC	BB	BB	BB	BB	BBB	BBB	BBB
T	BB	A	A	A	A	A	A	AA	CCC	BBB	BBB	A	BBB	A	A	A
TRV	BB	A	A	A	A	A	A	AA	CCC	BBB	BBB	A	BBB	A	A	AA
UNH	BB	A	A	A	BBB	A	A	AA	B	BBB	BBB	A	A	BBB	A	AA
UTX	BB	A	A	A	BB	A	A	AA	CCC	BBB	A	A	BBB	BBB	A	A
VZ	BB	A	A	AA	BBB	A	A	AA	CCC	BBB	A	A	A	A	A	AA
WMT	BB	A	A	AA	A	A	A	AA	CCC	BBB	BBB	A	BBB	A	A	AA
XOM	CCC	BBB	BBB	A	CCC	A	A	A	CCC	BB	BBB	BBB	BBB	A	A	A
Mean	BB	A	A	A	BB	A	A	AA	CCC	BBB	BBB	A	BBB	A	A	A

Table 2
Discrete Volatility Function

For each stock we run the cross-sectional regression $\sigma(M, T) = \beta_0 + \beta_1 M + \beta_2 M^2 + \beta_3 T + \beta_4 T^2 + \beta_5 XT$ at each month-end for the period August, 31 2005 to June, 30 2014. In total, 107 regressions are run for each stock and we report the mean of each coefficient and the standard deviations around this mean. M is defined as $\log(\text{Strike Price}/\text{Forward Price})$ and T is defined as the \log of the time to expiration. All coefficients are in percentages.

Ticker	Mean coefficients (standard deviations)											Goodness of fit		
	C		M		M ²		T		T ²		MT		R ²	
	(1)	(2)	(3)	(4)	(5)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
AA	38.7	(13.9)	-10.1	(5.6)	6.0	(5.9)	-0.5	(2.6)	0.5	(1.6)	3.4	(3.5)	0.90	(0.13)
AXP	32.5	(14.9)	-19.3	(5.6)	10.2	(6.2)	0.1	(2.7)	-0.4	(1.3)	7.9	(3.1)	0.98	(0.01)
BA	29.6	(7.2)	-16.3	(4.5)	10.6	(4.2)	0.7	(1.5)	-0.1	(1.0)	6.5	(2.5)	0.99	(0.01)
BAC	38.8	(22.0)	-17.3	(7.2)	10.9	(9.9)	0.2	(4.5)	0.4	(1.8)	6.8	(5.6)	0.96	(0.06)
CAT	32.4	(8.9)	-15.9	(5.9)	8.0	(3.9)	0.4	(1.6)	0.0	(1.1)	6.3	(2.4)	0.98	(0.02)
CSCO	30.2	(6.8)	-14.6	(5.3)	12.0	(4.8)	0.7	(1.3)	-0.3	(0.8)	5.0	(2.9)	0.96	(0.03)
CVX	25.5	(6.3)	-19.2	(6.8)	14.7	(7.0)	0.6	(1.5)	-0.6	(1.4)	7.7	(3.1)	0.98	(0.01)
DD	27.0	(8.0)	-17.8	(5.6)	14.4	(7.0)	0.5	(1.7)	-0.5	(0.9)	7.2	(2.8)	0.98	(0.02)
DIS	27.6	(7.3)	-18.1	(5.3)	13.9	(6.5)	0.6	(1.7)	-0.6	(0.9)	6.6	(2.6)	0.97	(0.02)
GE	28.2	(13.1)	-17.0	(8.8)	16.5	(12.9)	1.0	(1.6)	-0.1	(1.4)	8.9	(6.2)	0.93	(0.13)
HD	29.0	(8.8)	-19.0	(5.4)	13.5	(7.0)	0.8	(1.6)	-0.6	(1.1)	6.7	(2.8)	0.98	(0.02)
HPQ	32.7	(6.4)	-14.0	(4.4)	9.2	(3.6)	0.5	(1.4)	-0.2	(1.0)	5.1	(2.1)	0.98	(0.04)
IBM	23.2	(6.0)	-18.4	(5.6)	12.5	(4.9)	0.7	(1.4)	-0.5	(1.0)	7.5	(2.9)	0.98	(0.03)
INTC	29.9	(7.3)	-13.8	(5.5)	13.1	(6.4)	0.8	(1.3)	0.0	(0.8)	4.9	(2.9)	0.95	(0.08)
JNJ	18.4	(4.2)	-19.3	(5.6)	19.6	(7.6)	1.1	(1.3)	-0.6	(0.9)	8.1	(3.0)	0.98	(0.02)
JPM	32.2	(13.7)	-22.2	(5.8)	14.1	(5.9)	0.2	(2.4)	0.1	(1.5)	8.0	(3.2)	0.98	(0.02)
KO	19.7	(5.3)	-18.1	(5.3)	17.8	(7.7)	1.2	(1.1)	-0.5	(0.9)	7.4	(3.0)	0.97	(0.02)
MCD	22.3	(5.5)	-18.9	(5.6)	17.7	(7.6)	0.9	(1.2)	-0.5	(1.0)	6.7	(2.6)	0.98	(0.02)
MMM	23.5	(5.6)	-18.5	(6.4)	11.6	(7.5)	0.8	(1.4)	-0.4	(1.0)	8.5	(2.9)	0.98	(0.02)
MRK	26.2	(6.9)	-17.6	(6.0)	17.0	(8.9)	0.8	(1.8)	-0.3	(1.4)	7.3	(3.1)	0.97	(0.02)
MSFT	26.5	(7.7)	-16.5	(4.3)	15.1	(6.4)	1.0	(1.3)	0.1	(0.8)	5.3	(2.5)	0.97	(0.02)
PFE	25.6	(6.6)	-15.8	(7.0)	17.3	(11.9)	1.1	(1.1)	-0.3	(1.2)	5.7	(3.1)	0.93	(0.13)
PG	19.9	(4.9)	-21.1	(6.4)	17.8	(10.0)	1.1	(1.1)	-0.8	(0.9)	9.6	(3.7)	0.97	(0.05)
T	23.4	(7.3)	-19.4	(6.7)	25.1	(15.8)	1.1	(1.5)	-0.4	(1.3)	8.3	(3.6)	0.95	(0.10)
TRV	25.9	(9.0)	-20.3	(5.1)	13.5	(6.8)	0.2	(2.4)	-0.7	(1.3)	8.2	(2.2)	0.98	(0.02)
UNH	32.1	(10.0)	-14.7	(4.9)	9.1	(6.1)	0.5	(1.7)	-0.3	(1.4)	5.8	(2.9)	0.98	(0.02)
UTX	25.4	(6.2)	-20.6	(5.5)	13.3	(6.0)	1.1	(1.5)	-0.4	(1.2)	8.8	(2.8)	0.98	(0.01)
VZ	23.5	(6.4)	-17.7	(5.2)	22.6	(9.2)	1.3	(1.4)	-0.6	(1.1)	7.5	(3.5)	0.97	(0.02)
WMT	21.5	(5.6)	-18.0	(5.3)	16.8	(8.2)	1.2	(0.8)	-0.6	(1.0)	7.1	(2.9)	0.98	(0.01)
XOM	24.0	(6.1)	-20.2	(6.5)	15.9	(7.3)	0.8	(1.6)	-0.6	(1.0)	8.2	(3.5)	0.97	(0.02)

Table 3

Estimation results: Relative Root Mean Squared Error

We calibrate the 3 parameters $\{a_0, a_1 \text{ and } b\}$ so that the mean squared error (MSE) is minimized. The MSE is defined as: $MSE = \frac{1}{4} \sum_{k=1}^4 (CDS_{model} - CDS_{market})^2$. We start with the calibration for the Das and Hanouna model (DH) for the full period (August 2005-June 2014). Then we calibrate for the implied model where we gradually increase the number of variables from the discrete volatility function: $\sigma(M, T) = \beta_0 + \beta_1 M + \beta_2 M^2 + \beta_3 T + \beta_4 T^2 + \beta_5 MT$. In the constant volatility model (CV) we only include β_0 , in the term structure model we include β_0, β_3 and β_4 , in the skew (SK) we include β_0, β_1 and β_2 and finally in then volatility surface mode (VS) we include all 5 β 's. The full period has 107 monthly observations, and to evaluate the performance in different subperiod, we split the sample in three, where the first two subsamples have 35 observations and the last subsample has 36 observations. We report the relative root mean square error where take the standard deviation of the MSE and divide it with the average of the CDS spreads (similar to DH)

Ticker	Full period					August 2005 - July 2008					August 2008- June 2011					July 2011- June 2014				
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)
	DH	CV	TS	SK	VS	DH	CV	TS	SK	VS	DH	CV	TS	SK	VS	DH	CV	TS	SK	VS
AA	2.1	3.3	2.4	2.8	2.4	1.3	1.8	1.9	1.8	1.8	1.9	2.4	1.9	1.9	1.9	3.2	5.7	3.4	4.7	3.4
AXP	1.5	2.4	1.5	2.4	2.1	2.6	3.5	2.7	3.1	2.4	0.9	2.1	1.0	2.5	2.0	0.9	1.5	0.8	1.6	1.7
BA	1.7	2.0	1.6	1.7	1.6	1.4	1.5	1.3	1.3	1.1	1.8	2.1	1.7	2.5	1.9	2.0	2.3	1.7	1.2	1.9
BAC	1.3	1.6	1.6	1.5	1.9	2.2	2.4	2.7	1.9	2.8	1.1	1.4	1.2	1.4	1.1	0.6	0.9	1.0	1.1	1.9
CAT	1.8	2.1	1.7	2.1	2.2	2.1	2.0	1.9	1.8	1.7	1.1	1.3	1.2	2.3	1.9	2.3	2.9	2.1	2.2	2.9
CSCO	3.2	2.9	2.7	3.2	3.0	5.9	5.2	5.1	4.3	4.1	2.0	1.9	1.7	2.7	2.1	1.6	1.5	1.2	2.5	2.8
CVX	2.7	2.5	2.4	2.4	2.7	3.5	3.5	3.5	3.2	3.1	2.2	2.3	1.9	2.7	2.6	2.2	1.8	1.9	1.5	2.3
DD	1.8	2.0	1.8	2.4	2.3	1.2	1.6	1.7	1.6	1.7	2.4	2.5	2.4	3.0	3.1	1.8	2.0	1.4	2.4	2.3
DIS	1.9	2.0	2.0	3.0	2.7	2.9	3.2	3.0	2.3	2.1	1.9	2.0	2.0	3.1	3.1	0.7	0.7	0.9	3.7	2.7
GE	3.1	4.0	3.4	3.3	3.3	4.8	6.8	5.8	5.3	5.8	2.4	2.5	2.2	2.3	1.9	1.9	2.7	2.2	2.2	2.3
HD	1.9	2.0	1.7	2.0	1.8	1.4	1.8	1.6	1.7	1.5	1.4	1.5	1.3	2.5	1.9	2.9	2.7	2.1	1.9	1.9
HPQ	1.5	2.0	1.7	1.9	1.7	1.5	2.0	1.8	1.5	1.4	1.8	2.0	1.8	2.2	2.1	1.2	2.0	1.6	2.0	1.7
IBM	2.2	2.5	2.2	2.1	2.2	1.8	2.5	2.3	2.3	2.0	2.8	2.8	2.6	3.0	2.9	2.0	2.3	1.6	1.2	1.7
INTC	1.8	1.8	1.7	2.0	1.8	2.3	2.5	2.3	2.4	1.9	1.8	1.7	1.5	1.7	1.2	1.4	1.3	1.3	2.0	2.4
JNJ	2.4	2.5	2.3	1.8	2.2	2.8	2.9	3.0	2.5	2.4	2.6	2.7	2.6	2.2	2.6	2.0	2.0	1.3	0.8	1.5
JPM	1.3	1.5	1.4	2.0	1.9	1.7	1.9	1.8	1.9	1.6	1.6	1.8	1.4	2.4	1.9	0.7	1.0	0.9	1.7	2.2
KO	2.0	2.4	2.3	1.8	2.2	2.4	2.9	3.1	2.3	2.9	2.1	2.6	2.3	2.2	2.5	1.6	1.6	1.3	1.0	1.2
MCD	2.3	2.1	2.0	2.0	2.4	3.2	3.1	3.0	2.5	2.2	1.3	1.2	1.2	2.0	2.5	2.2	2.0	1.7	1.4	2.5
MMM	1.6	1.6	1.7	2.1	2.7	1.9	2.1	2.1	1.8	1.7	2.0	2.0	1.8	2.2	2.3	0.8	0.6	1.2	2.3	3.9
MRK	1.6	1.6	1.5	2.5	2.1	1.5	1.7	1.6	2.4	2.0	2.2	2.4	1.9	3.5	2.5	0.9	0.8	0.8	1.8	1.7
MSFT	2.5	2.5	2.2	2.0	1.9	1.8	1.5	1.5	1.8	1.0	4.1	4.1	3.7	2.4	3.1	1.5	1.9	1.6	1.9	1.7
PFE	1.7	1.6	1.6	1.8	2.1	1.4	1.4	1.5	1.6	2.3	1.7	1.7	1.8	2.2	2.1	2.1	1.7	1.5	1.7	2.0
PG	2.0	2.2	1.7	1.8	2.1	1.8	2.1	1.9	2.0	1.7	2.1	2.1	1.8	2.0	2.1	2.1	2.2	1.4	1.4	2.4
T	1.8	1.8	1.6	2.4	2.7	1.3	1.9	1.7	2.8	3.4	3.1	3.0	2.7	2.6	3.0	0.9	0.6	0.5	1.9	1.6
TRV	1.7	2.0	1.6	1.9	1.9	2.2	2.5	2.6	2.6	2.4	1.8	2.0	1.4	1.9	1.4	1.1	1.5	0.9	1.2	1.8
UNH	2.0	2.6	2.1	1.5	1.9	2.2	2.9	2.8	2.4	2.6	0.8	1.2	1.0	1.2	1.0	3.1	3.6	2.5	1.0	2.2
UTX	1.5	1.7	1.6	2.3	3.0	1.6	2.2	2.1	2.1	2.0	1.8	1.8	1.7	2.9	2.6	1.1	1.1	1.0	2.0	4.3
VZ	2.4	2.6	2.2	2.8	2.4	4.4	4.6	3.8	3.5	3.1	2.2	2.5	2.3	3.8	2.6	0.6	0.7	0.4	1.2	1.5
WMT	1.6	1.6	1.5	1.7	2.2	1.5	1.8	1.7	1.7	1.8	1.8	1.9	1.6	2.0	2.0	1.5	1.1	1.2	1.4	2.7
XOM	2.9	2.6	2.5	3.1	3.5	3.2	2.7	2.4	1.8	1.6	3.9	3.7	3.3	4.0	3.7	1.5	1.3	1.7	3.5	5.1
Mean	2.0	2.2	1.9	2.2	2.3	2.3	2.6	2.5	2.3	2.3	2.0	2.2	1.9	2.4	2.3	1.6	1.8	1.4	1.9	2.3

Table 4

Estimation results: Mean Model Parameters

We report the mean estimated model parameters $\{a_0, a_1 \text{ and } b\}$ for the full period August 2005 till June 2014 for the 5 different model specifications. The first specification is the DH model and the four subsequent models are based on different specifications of the discrete volatility function: $\sigma(M, T) = \beta_0 + \beta_1 M + \beta_2 M^2 + \beta_3 T + \beta_4 T^2 + \beta_5 MT$. In the constant volatility model (CV) we only include β_0 , in the term structure model we include β_0, β_3 and β_4 , in the skew (SK) we include β_0, β_1 and β_2 and finally in then volatility surface mode (VS) we include all five β 's. The full period has 107 monthly observations and we also report the mean of the stock price over this period

Ticker	DH Model			Constant Volatility (CV)			Term Structure (TS)			Skew (SK)			Volatility Surface (VS)			Stock Price
	(1) a_0	(2) a_1	(3) b	(4) a_0	(5) a_1	(6) b	(7) a_0	(8) a_1	(9) b	(10) a_0	(11) a_1	(12) b	(13) a_0	(14) a_1	(15) b	(16) S
AA	2.7	-92.2	1.40	3.6	-245.8	1.53	3.7	-221	1.46	3.5	-210.9	1.52	3.6	-214.9	1.53	18.9
AXP	4.8	-180.5	0.90	5.3	-626.1	1.09	5.6	-751.6	1.11	5.8	-338.3	1.00	5.7	-529.9	1.09	52.2
BA	4.2	-370.8	1.01	5.1	-762.6	1.11	5.1	-868	1.11	5.4	-422.5	1.04	5.0	-781.4	1.12	77.7
BAC	2.2	-139.5	1.49	2.6	-256.1	1.60	2.8	-365.4	1.57	3.2	-93.8	1.40	3.0	-77.5	1.33	24.1
CAT	4.2	-328	1.03	4.7	-798.3	1.08	4.9	-821.8	1.06	5.5	-577.5	1.02	5.2	-674.5	1.05	75.2
CSCO	2.5	-261.1	1.63	2.7	-584.1	1.69	2.9	-668.6	1.73	3	-228.9	1.62	3.0	-207.5	1.59	21.6
CVX	3.8	-571.3	1.10	4.4	-1022.4	1.15	4.4	-1351.9	1.19	5.7	-366.2	1.03	4.9	-347.3	1.21	89.3
DD	4.1	-669.5	1.29	5.2	-1165.2	1.34	5.3	-1346.1	1.36	6	-333.9	1.06	5.7	-312.9	1.13	45.6
DIS	3.5	-560.1	1.41	4.2	-1157.6	1.51	4.3	-1352.4	1.52	4.6	-621.8	1.41	4.5	-252	1.35	39.4
GE	4.2	-216.6	1.34	5.5	-622.1	1.40	5.8	-430.7	1.39	5	-312.8	1.31	5.5	-336.3	1.23	24.6
HD	3.6	-282.2	1.11	4.1	-595.9	1.24	3.8	-743.4	1.29	4.9	-231.1	1.06	4.0	-244.7	1.10	42.5
HPQ	3.0	-259.9	1.37	3.6	-524.1	1.39	3.7	-553	1.42	4	-327.6	1.38	4.0	-632.6	1.44	35.4
IBM	5.7	-446.8	0.84	6.6	-1186.9	0.97	6.4	-1329.2	1.00	6.9	-619.5	0.89	6.2	-1240.9	0.98	138.5
INTC	2.1	-206.9	1.68	2.4	-503.8	1.80	2.5	-471.7	1.80	2.6	-226.2	1.77	2.8	-162.2	1.46	21.9
JNJ	4.8	-546.3	1.15	5.8	-1332	1.21	5.9	-1648	1.29	6.2	-592.5	1.12	5.8	-777.1	1.21	67.5
JPM	3.9	-415.3	1.23	4.8	-894.3	1.24	4.9	-755	1.22	5.6	-175.4	1.12	5.3	-169	1.06	43
KO	5.2	-568.4	1.24	6.1	-999.7	1.30	6.1	-1203.4	1.35	6.2	-601.6	1.27	6.1	-1493.9	1.36	51.6
MCD	4.4	-504.1	1.11	4.9	-1200	1.22	4.9	-1252	1.26	5.6	-395.1	1.05	4.8	-341.5	1.07	68.6
MMM	4.3	-890.5	1.18	5.1	-1780.8	1.24	5.1	-2060.3	1.25	5.7	-1686.5	1.24	5.2	-1732.1	1.26	86.6
MRK	3.3	-774.4	1.48	4.1	-1302.4	1.49	4.3	-1468.7	1.45	4.7	-479.4	1.44	5.0	-448	1.19	39.7
MSFT	2.2	-190	1.49	2.5	-434.9	1.59	2.5	-366.1	1.56	5.3	-242.2	1.28	2.4	-138.9	1.41	28.4
PFE	2.6	-560.6	1.59	3.5	-837.3	1.72	3.5	-855	1.71	5.2	-261	1.42	4.6	-208.2	1.19	22.4
PG	5.5	-664.7	1.08	6.8	-1455.2	1.14	7.1	-1380.9	1.16	7.1	-824.5	1.08	7.5	-991	1.12	65
T	3.2	-875.5	1.56	3.8	-1283	1.60	4.0	-1529.8	1.60	5.4	-236.2	1.22	5.7	-281	1.10	31.6
TRV	4.8	-342.2	1.01	6.1	-869.5	1.10	6.7	-992.1	1.12	6.4	-556.1	1.02	6.8	-694.7	1.05	57.1
UNH	4.2	-222	0.93	4.6	-552	1.10	4.5	-624.6	1.13	4.8	-363.7	1.04	4.3	-502	1.15	47.6
UTX	4.9	-608.2	1.08	5.5	-1271	1.18	5.9	-1736.2	1.19	6.4	-849.9	1.12	6.1	-1112	1.18	74.4
VZ	4.3	-364.1	1.18	4.7	-704.8	1.30	4.8	-884	1.33	6	-485.2	1.21	6.4	-412.6	1.20	37.8
WMT	5.1	-1028	1.22	6.6	-1941	1.26	6.8	-2133.2	1.25	6.4	-937.6	1.18	6.4	-1488.1	1.24	56.9
XOM	3.8	-483.5	1.18	4.1	-832.1	1.24	3.9	-795.5	1.24	4.2	-461.6	1.18	3.6	-409.9	1.18	78.5
Mean	3.9	-454.1	1.24	4.6	-924.7	1.33	4.7	-1032	1.34	5.2	-468.7	1.22	5.0	-573.8	1.22	52.1

Table 5

Estimation results: Two-year (cumulative) default probabilities

We report the two-year cumulative probability (δ_j^{cum}) as calculated by the equation (41):

$\delta_j^{cum} = \sum_{i=0}^j p_j^i * \delta_j^i + \sum_{i=0}^j (1 - p_j^i)$. The probability p_j^i indicates the probability of reaching a state from all possible paths in the binomial tree that is governed both by the node probability p_j^i as by the survival probability z_j^i .

Ticker	Full period					August 2005 - July 2008					August 2008- June 2011					July 2011- June 2014				
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)
AA	9.4	8.3	8.9	8.0	8.3	3.3	1.3	1.8	1.1	1.3	16.6	12.6	12.5	12.1	11.4	8.4	11.1	12.3	11.0	12.3
AXP	10.7	5.4	4.3	8.3	6.3	9.2	3.9	3.5	5.2	3.6	16.4	10.3	7.4	15.1	10.4	6.4	2.0	1.8	4.6	4.8
BA	4.5	2.3	2.1	3.4	3.5	1.1	0.8	0.8	1.2	0.9	6.9	4.0	3.6	4.2	5.8	5.6	2.1	1.9	4.7	3.8
BAC	7.1	4.7	5.8	6.4	8.3	1.1	0.8	1.3	2.9	2.2	14.9	8.6	10.4	10.9	14.8	5.3	4.8	5.5	5.5	8.0
CAT	5.5	2.4	2.5	4.9	5.1	1.0	0.9	0.8	1.0	0.9	9.5	4.1	4.8	9.6	10.5	6.0	2.2	2.0	4.2	3.8
CSCO	2.8	1.5	1.4	3.6	4.4	1.3	0.8	0.8	1.0	0.8	3.7	2.5	2.4	6.2	8.2	3.5	1.2	1.0	3.7	4.2
CVX	2.1	1.2	1.5	3.4	3.7	0.6	0.4	0.4	0.7	0.7	3.7	1.9	2.9	4.5	6.5	1.8	1.2	1.1	5.0	3.8
DD	2.9	2.6	2.5	4.4	4.3	1.0	0.7	0.9	1.0	1.2	4.0	5.7	5.3	7.4	8.1	3.8	1.3	1.2	4.7	3.7
DIS	2.5	1.1	1.1	3.1	4.0	1.7	0.9	0.8	1.2	0.8	3.5	1.8	1.7	5.1	7.3	2.3	0.5	0.7	2.9	3.8
GE	6.6	4.4	4.8	5.7	6.9	2.1	2.0	1.8	1.7	2.6	13.6	8.6	9.1	11.2	12.6	4.2	2.5	3.4	4.0	5.6
HD	6.2	3.2	3.0	4.8	4.7	3.2	2.0	1.5	2.0	2.8	12.5	6.0	6.1	8.8	8.4	2.8	1.4	1.2	3.5	2.8
HPQ	4.8	2.2	2.0	3.3	3.5	0.9	0.6	0.7	0.9	0.7	8.2	3.2	2.9	5.4	6.1	5.4	2.8	2.6	3.8	3.7
IBM	2.6	1.3	1.2	2.4	1.6	2.1	1.1	1.0	1.1	1.0	2.6	1.4	1.3	2.5	1.8	3.2	1.5	1.3	3.7	1.9
INTC	3.5	2.0	2.2	3.6	4.5	0.7	0.5	0.9	0.6	1.4	6.6	4.6	4.1	6.0	6.9	3.2	0.8	1.5	4.2	5.3
JNJ	1.8	1.1	1.0	2.4	2.7	0.8	0.5	0.5	0.7	0.8	1.8	1.5	1.4	3.4	4.5	2.7	1.2	1.2	3.0	3.0
JPM	5.3	3.5	3.7	5.3	5.9	1.2	0.9	1.2	2.0	2.7	11	7.5	7.8	9.7	9.0	3.8	2.1	2.1	4.1	5.9
KO	3.6	2.2	2.2	2.6	2.7	1.0	0.9	0.8	0.7	0.8	7.6	4.3	4.3	4.6	5.1	2.3	1.5	1.7	2.5	2.2
MCD	2.7	1.2	1.3	3.3	3.5	1.4	0.8	0.9	1.9	1.7	4.9	2.0	2.1	5.4	6.6	1.8	0.9	0.9	2.7	2.2
MMM	2.2	1.1	1.2	2.1	2.5	1.1	0.7	0.6	0.9	0.6	3.0	1.9	2.2	3.5	4.1	2.6	0.7	0.7	2.0	2.9
MRK	2.1	1.1	1.6	3.4	5.2	0.5	0.4	0.5	0.9	0.7	2.1	1.5	2.4	6.0	10	3.6	1.4	1.9	3.4	5.0
MSFT	3.4	1.6	2.1	4.2	4.0	1.5	0.6	0.9	1.5	0.8	4.0	3.0	3.0	6.3	6.6	4.6	1.2	2.4	4.8	4.5
PFE	2.8	1.5	1.9	5.1	4.6	0.3	0.3	0.5	1.9	1.0	4.9	2.8	3.7	8.0	7.5	3.1	1.4	1.6	5.3	5.2
PG	2.8	1.7	1.9	3.0	3.8	1.1	0.7	0.7	1.3	0.9	4.2	2.6	2.9	5.0	5.6	3.1	1.7	2.3	2.7	4.8
T	2.1	1.0	1.4	3.9	4.7	1.9	0.7	0.8	1.4	0.7	3.2	1.5	2.5	5.9	6.7	1.3	0.7	1.0	4.5	6.6
TRV	4.3	2.4	3.2	3.1	4.1	1.3	0.9	1.1	1.6	1.4	6.2	3.9	5.7	3.9	6.0	5.3	2.3	2.7	3.9	5.0
UNH	6.6	2.8	2.9	4.1	3.6	2.1	1.3	1.2	1.8	2.1	13.2	5.1	5.2	6.9	5.5	4.4	2.2	2.3	3.7	3.1
UTX	3.4	1.7	1.8	2.6	4.3	1.8	0.9	1.1	1.4	0.6	4.6	2.5	2.7	4.2	7.2	3.9	1.5	1.5	2.3	5.2
VZ	4.7	2.0	3.2	3.7	5.2	3.0	1.6	1.5	1.8	1.7	5.5	2.2	5.6	5.1	10.3	5.6	2.1	2.4	4.1	3.5
WMT	2.2	1.2	1.7	2.1	2.4	1.2	0.6	0.8	1.5	0.6	3.8	1.9	2.9	2.7	4.1	1.5	1.0	1.5	2.2	2.6
XOM	2.0	1.3	1.8	2.2	2.6	1.6	0.7	1.7	1.6	1.7	3.6	2.4	3.0	3.2	4.1	0.8	0.7	0.5	1.9	1.9
Mean	4.1	2.3	2.5	3.9	4.4	1.7	1.0	1.1	1.5	1.3	6.9	4.1	4.4	6.4	7.4	3.7	1.9	2.1	4.0	4.4

Table 6

Estimation results: Two-year recovery rates

We estimate the two-year recovery rates based on the equation:

$R_j^{cum} = \sum_{i=0}^j p_j^i * R_j^i + \sum_{i=0}^j (1 - p_j^i) * R_{j-1}^{cum}$. The implied recovery rate for period j at time zero is equal to the recovery rates in each node times the probability of reaching that node (from all paths) plus the recovery rate from the previous layer in the three times the probability of default.

Ticker	Full period					August 2005 - July 2008					August 2008- June 2011					July 2011- June 2014				
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)
AA	DH	CV	TS	SK	VS	DH	CV	TS	SK	VS	DH	CV	TS	SK	VS	DH	CV	TS	SK	VS
AA	73.4	67.9	70.6	70.3	70.7	74.6	72.1	75.4	73.6	73.4	68.6	57.9	60.3	60.2	59.6	77.1	73.9	76.2	77.4	79.4
AXP	81.7	70.9	69.8	79.2	75.9	87.8	75.7	77.9	79.6	77.6	71.9	62.2	58.8	73.0	65.7	85.4	75.0	72.7	85.2	84.5
BA	78.0	71.9	71.3	76.6	77.5	80.0	77.2	76.8	80.8	78.8	64.9	57.8	58.6	61.2	68.8	89.4	80.9	78.8	88.1	85.3
BAC	64.9	60.4	64.7	70.1	75.4	69.4	66.5	68.1	76.4	74.0	68.9	58.5	65.7	70.0	79.0	56.2	55.9	60.1	63.8	73.0
CAT	76.7	70.6	70.9	77.6	78.2	78.6	78.5	77.1	79.0	78.6	65.2	56.0	60.2	71.0	76.6	86.5	77.4	75.5	83.1	79.3
CSCO	71.3	65.9	65.3	72.6	76.1	78.7	69.8	69.6	75.2	72.6	60.2	57.4	56.9	66.9	78.2	75.2	70.7	69.5	75.6	77.5
CVX	73.4	70.1	69.4	77.0	77.5	72.4	70.6	70.2	71.6	70.7	63.1	58.4	60.7	68.3	76.5	85.2	81.7	77.6	91.3	85.6
DD	73.8	71.2	71.8	80.6	78.9	75.5	72.8	74.2	76.2	77.0	62.1	63.4	65.6	83.4	83.0	84.1	77.5	75.7	82.2	76.6
DIS	76.3	71.8	71.7	76.3	84.6	87.5	82.9	81.5	84.0	82.4	68.9	64.4	64.4	72.5	86.7	72.4	68.1	69.3	72.3	84.6
GE	66.4	59.7	62.8	67.6	71.8	69.7	70.9	71.0	72.5	75.0	63.7	48.6	52.6	60.6	66.7	65.9	59.6	64.9	69.8	73.9
HD	79.1	73.1	71.8	77.8	79.3	77.3	73.4	71.0	75.7	78.8	69.8	62.6	64.8	75.9	79.7	90.4	83.7	79.9	81.9	79.5
HPQ	76.6	72.1	70.7	75.9	74.7	79.2	76.4	77.3	78.0	78.0	76.9	70.1	66.1	79.2	77.3	73.7	69.7	68.8	70.3	68.7
IBM	84.0	77.8	76.6	82.5	80.0	87.4	81.8	81.5	84.1	84.1	73.7	68.1	67.1	71.6	71.3	91.1	83.7	81.3	92.1	84.7
INTC	74.4	66.6	70.0	75.9	80.5	72.9	71.1	73.1	70.4	72.9	79.2	62.9	69.5	77.0	82.9	71.0	65.8	67.4	80.4	85.9
JNJ	79.5	74.7	72.5	82.4	80.8	82.5	82.4	82.6	83.8	84.3	64.9	58.5	53.9	72.1	69.9	91.3	83.6	81.3	91.5	88.2
JPM	68.8	65.3	68.2	74.5	79.6	71.6	70.2	72.0	75.1	83.2	68.4	63.5	69.7	75.5	77.1	66.3	62.1	62.7	72.7	78.4
KO	80.7	76.4	75.9	79.7	79.4	77.4	77.8	77.8	79.5	82.4	76.8	68.8	68.9	73.1	72.0	87.9	82.9	81.1	86.7	83.9
MCD	82.6	76.2	74.6	85.6	85.3	85.3	81.4	81.4	85.3	83.4	73.5	63.9	61.8	81.1	84.4	89.3	83.6	80.8	90.6	88.1
MMM	78.2	72.0	70.5	76.0	74.4	86.6	82.0	81.8	82.9	80.5	69.2	60.3	58.4	70.2	64.0	78.7	73.7	71.3	75.0	78.7
MRK	68.6	65.3	66.9	72.1	77.3	70.8	69.8	70.5	72.1	71.9	56.4	53.0	56.0	67.6	82.6	79.0	73.5	74.4	76.7	77.5
MSFT	75.4	70.6	72.7	66.7	79.5	80.9	70.8	72.9	55.6	72.4	73.8	74.6	74.5	72.3	83.5	71.6	66.3	70.8	72.5	82.7
PFE	70.7	67.2	69.0	52.1	80.5	66.3	67.0	68.5	41.5	68.3	60.5	54.8	61.0	54.1	84.0	85.6	80.2	77.7	60.8	89.4
PG	79.8	74.2	74.8	80.6	80.6	82.9	77.2	77.3	81.9	78.3	69.1	63.5	63.4	74.5	74.1	87.5	82.0	83.9	85.6	89.6
T	69.4	63.7	64.8	74.7	78.6	77.2	72.6	73.3	68.8	64.3	64.2	55.4	59.4	78.5	83.3	66.7	63.1	61.5	77.0	88.4
TRV	76.0	68.4	69.7	75.4	76.9	76.0	72.0	73.5	78.4	78.0	65.0	55.2	58.7	63.1	67.4	87.3	78.2	77.3	85.1	85.7
UNH	82.5	71.9	72.1	78.6	75.7	82.3	76.8	76.7	80.3	79.9	74.9	58.2	59.4	68.2	62.7	90.5	81.0	80.4	87.5	84.7
UTX	77.0	71.6	71.0	75.5	79.2	83.9	77.9	77.0	82.7	76.5	65.0	61.9	62.8	69.0	74.4	82.1	75.2	73.4	74.9	86.8
VZ	78.3	71.6	75.1	77.7	79.4	85.8	80.2	79.0	81.3	80.5	65.5	56.2	70.0	69.5	86.4	83.7	78.7	76.3	82.4	71.0
WMT	75.8	72.1	73.2	79.7	77.1	80.2	78.0	77.8	84.3	76.9	67.2	62.0	63.8	69.6	70.0	80.2	76.5	78.1	85.3	84.7
XOM	77.5	74.9	75.3	75.5	76.5	84.4	81.3	87.0	83.9	83.6	75.2	70.9	69.6	72.1	79.8	72.9	72.3	69.2	70.5	66.0
Mean	75.7	70.2	70.8	75.6	78.1	78.8	75.2	75.8	76.5	77.3	68.2	61.0	62.8	70.7	75.6	80.1	74.6	73.9	79.6	81.4

Figure 1 Volatility skew in the Das-Hanouna model

We start a binomial tree with a stock price S of 100 and we let the tree grow with a volatility 25% with either a time step of 0.5 or 0.05. The tree will have a length of 2 years. Then we vary the default probability in the tree by letting b vary from 0.5 to 100. In the resulting trees we calculate the value of the option at different moneyness points (-0.2, -0.1, 0, 0.1 and 0.2). The implied volatility of is then calculated by using the Black-Scholes formula

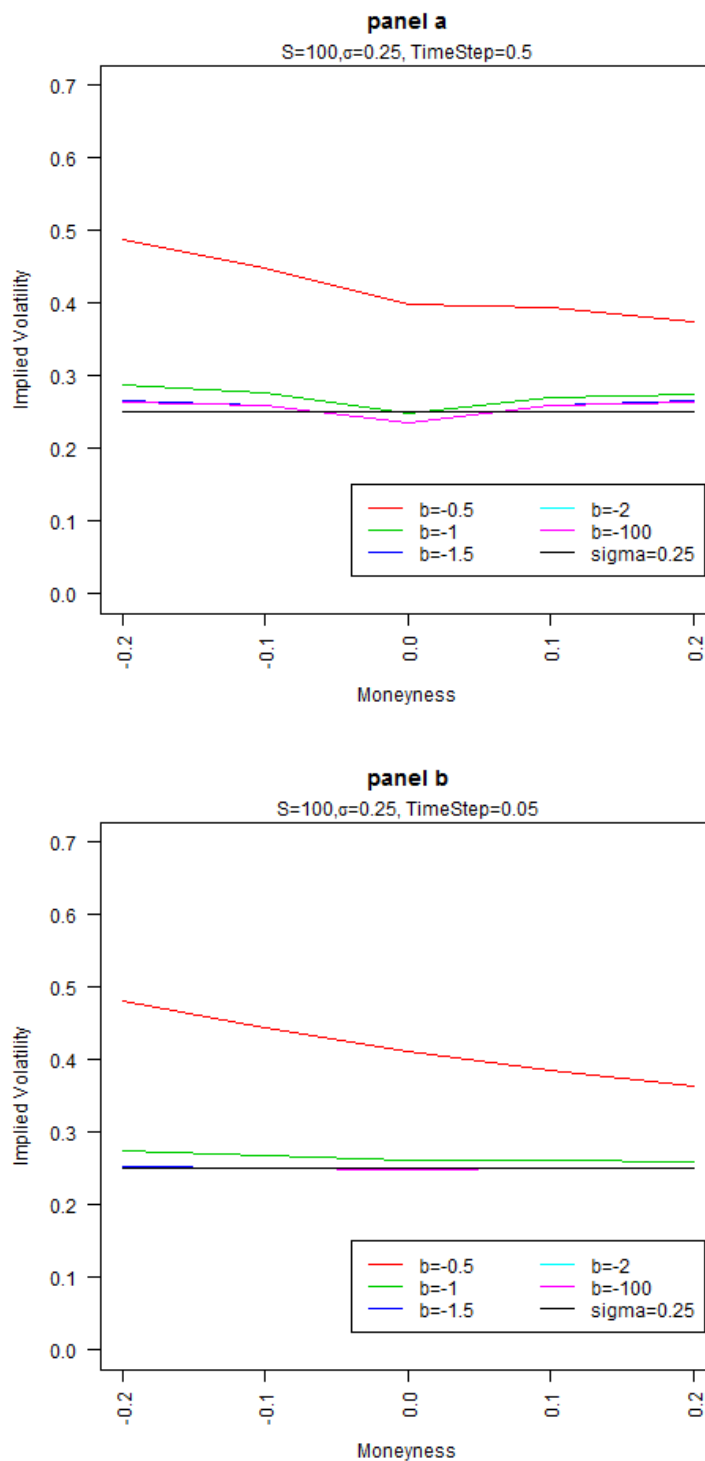


Figure 2 Representation of DJIA companies in the Markit CDS database

The Dow Jones Industrial Index consists of 30 companies and we check how many of these companies were priced in the Markit CDS database for the period January 2001 till June 2014. We checked for the three tenors we are interested in: half year, one year and two years plus we added a five-year tenor that is viewed by market participants as the most liquid CDS spread.

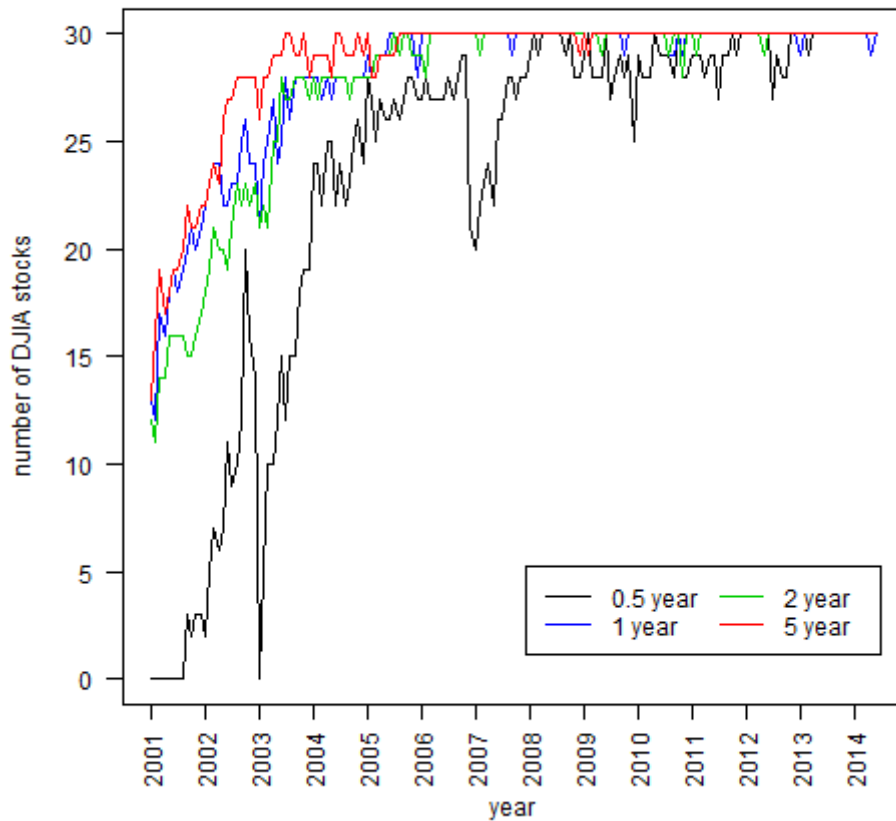


Figure 3 Re-Engineered Implied Volatilities

We calculate the two-year implied volatility for a specific date with the Black-Scholes formula with as input the 2-year binomial option prices from the DH, TS and VS models for the four exercise prices (moneyness) that we have in the fourth layer of the binomial tree. Then we interpolate this moneyness to -0.3, -0.1, 0.1 and 0.3. Then we run this for all dates for a company and take the average for the period. We do this for all 30 companies and report the average for these companies.

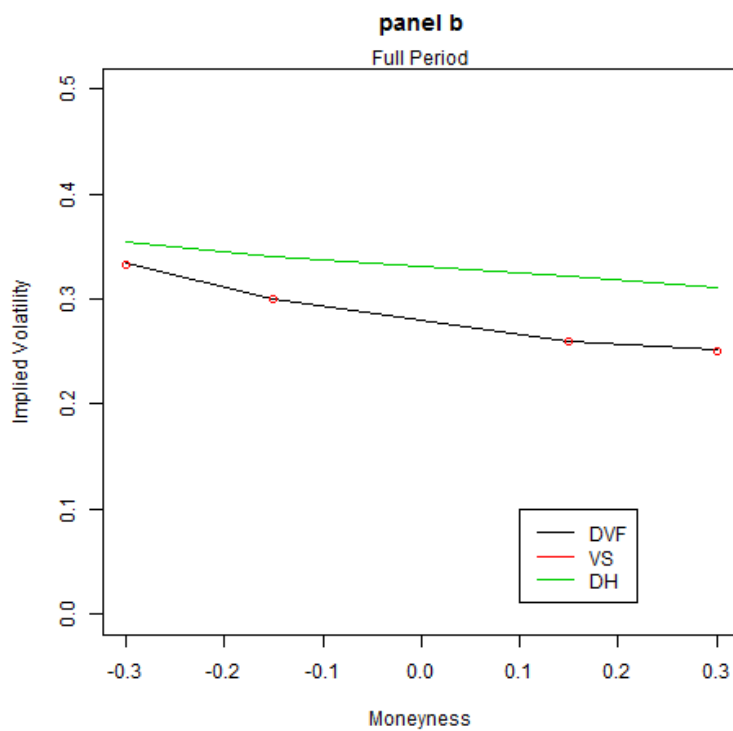
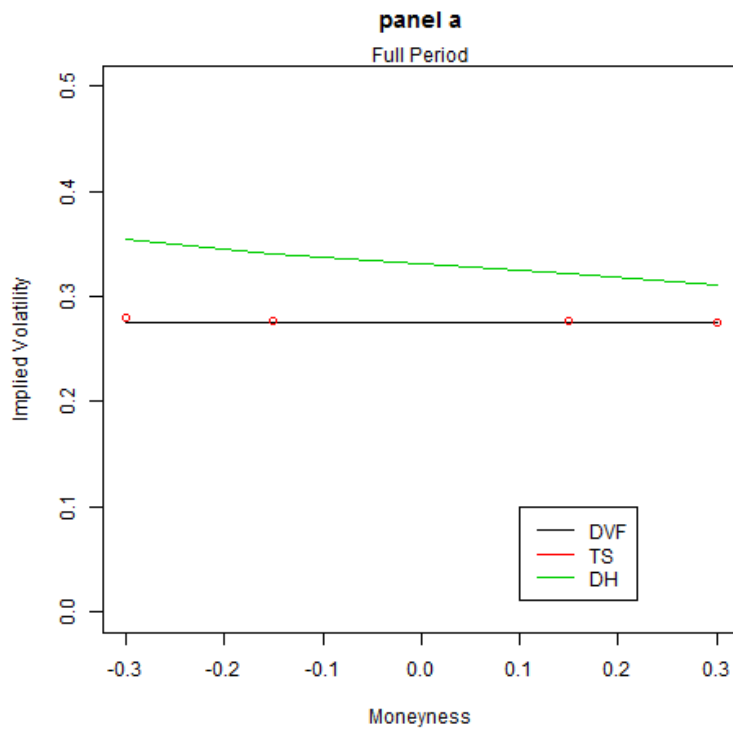


Figure 4 Marginal Recovery Rates

We simulate the recovery rates for two equations for the recovery rate. The first equation is as in Das and Hanouna (2009 equation (14)):

$$R_j^{margin} = \sum_{i=0}^j p_j^i * R_j^i \text{ and the second is our equation: } R_j^{margin} = \frac{\sum_{i=0}^j p_j^i * R_j^i}{\sum_{i=0}^j p_j^i}.$$

Then we simulate the recovery rates in two cases: (1) the recovery rate is set to a constant value of one with a_0 is 100 and a_1 set to zero and (2) the recovery rate is non-constant and dependent on the development of the stock price ($a_1=-200$). Additionally we have set interest rate to zero and b to one.

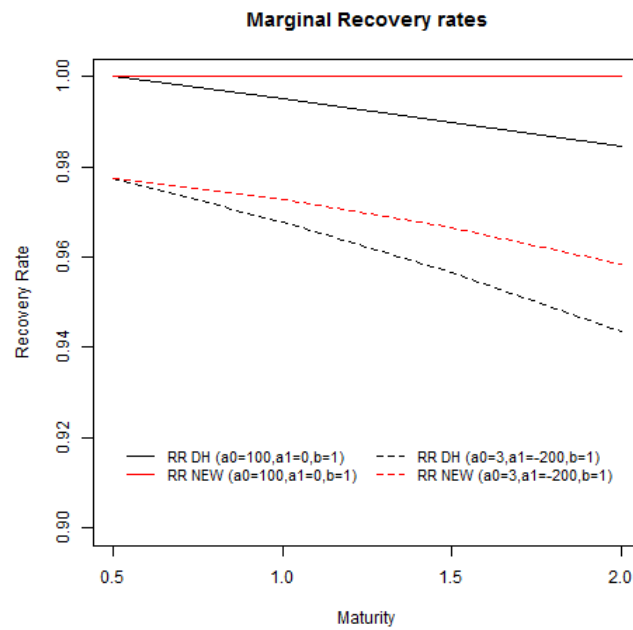


Figure 5 Term Structure of Forward Recovery Rates

For each company we run for the full period (August 2005 till June 2014) equation (43), $R_{j,j+1}^{forw} = \frac{\sum_{i=0}^j p_j^i * R_j^i}{\sum_{i=0}^j p_j^i}$, that gives the forward (or marginal) recovery rate for the four tenors (0.5, 1, 1.5 and 2 years). Then we average over all companies and dates to come up with the recovery rate for that specific tenor for of either the Das Hanouna (DH) model or the implied models with term structure (TS) and volatility surface (VS) specification.

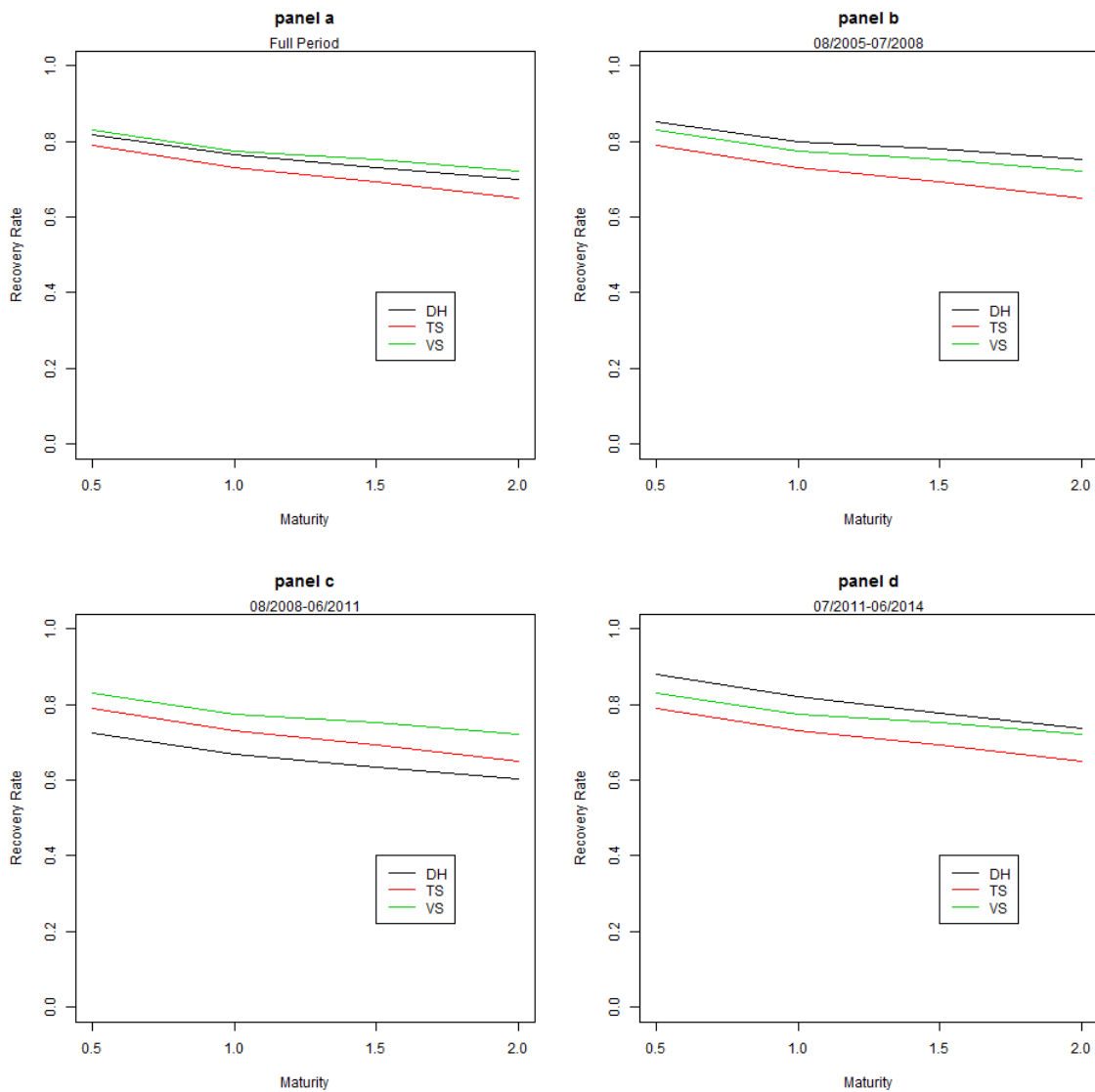
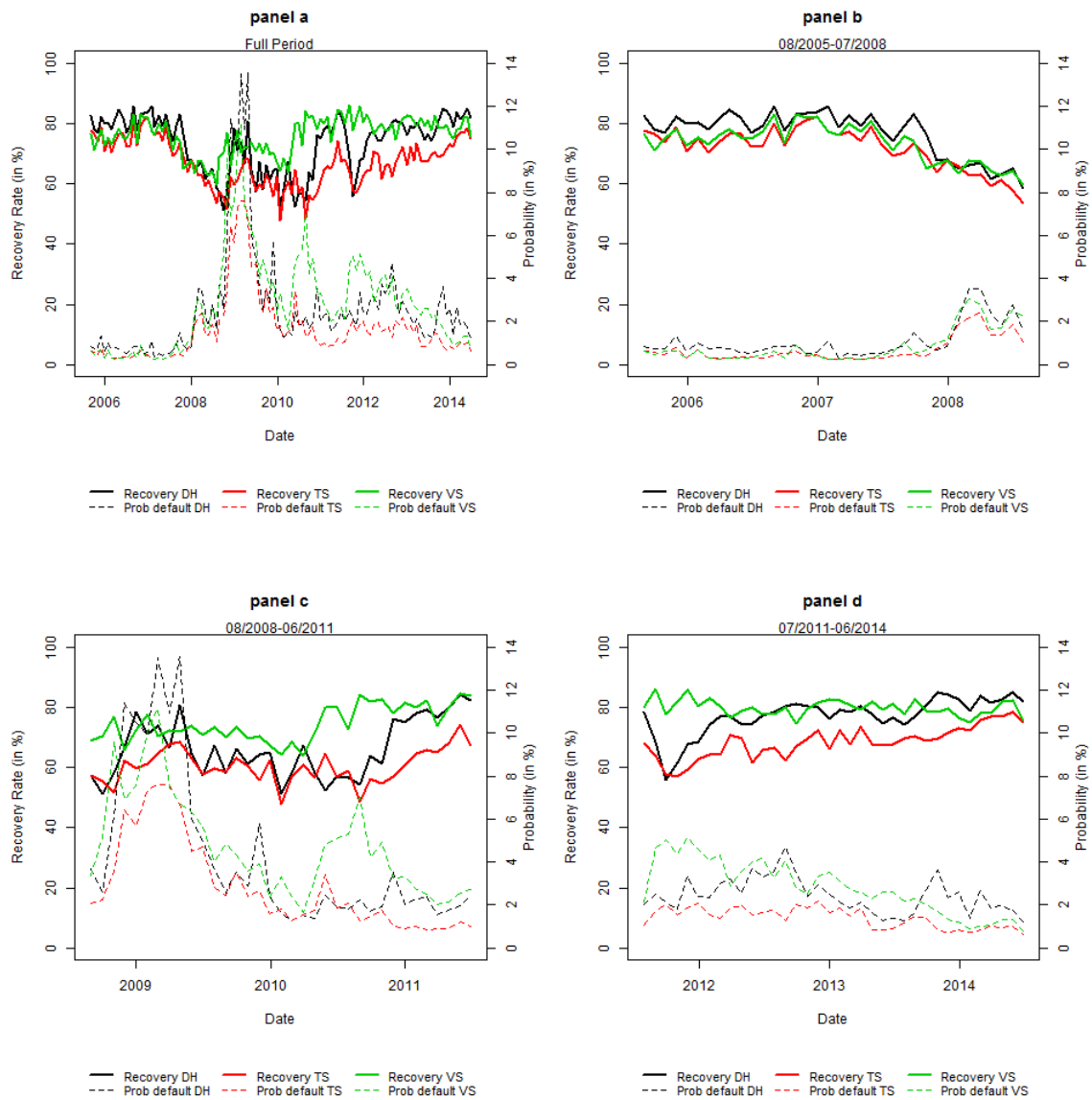


Figure 6 2-Year Default Probabilities and Recovery Rates

The time-series of recovery rates and default probabilities are shown for two different model specifications of the implied model and the Das and Hanouna model (DH). The Term Structure (TS) specification includes the term structure parameters while the Volatility Surface (VS) specification includes all parameters. Panels b to d show the different sub-periods.



Appendix A Intermediate steps to go from equation (15) to equation (16)

We start with the equation (15):

$$\begin{aligned} C[K_{j-1}^k, j] &= e^{-r_{j-1}h} (\lambda_{j-1}^{j-1} q_{j-1}^{j-1} z_{j-1}^{j-1} (S_j^j - K_{j-1}^k) + (\lambda_{j-1}^k q_{j-1}^k z_{j-1}^k \\ &\quad + \lambda_{j-1}^{k+1} (1 - q_{j-1}^{k+1}) z_{j-1}^{k+1} (S_j^{k+1} - K_{j-1}^k) \\ &\quad + \sum_{i=k+2}^{j-1} (\lambda_{j-1}^{i-1} q_{j-1}^{i-1} z_{j-1}^{i-1} + \lambda_{j-1}^i z_{j-1}^i (1 - q_{j-1}^i)) (S_j^i - K_{j-1}^k)) \end{aligned} \quad (\text{A.1})$$

then we bring $e^{-r\Delta t}$ to the left side of equation A.1, we separate the summation terms in two individual terms and separate the summation terms in two individual terms and change the counters of the last (summing) term from $\sum_{i=k+2}^{j-1}$ to $\sum_{i=k+3}^j$:

$$\begin{aligned} e^{r_{j-1}h} C[K_{j-1}^k, j] &= \left(\lambda_{j-1}^{j-1} q_{j-1}^{j-1} z_{j-1}^{j-1} (S_j^j - K_{j-1}^k) \right. \\ &\quad + \left(\lambda_{j-1}^k q_{j-1}^k z_{j-1}^k (S_j^{k+1} - K_{j-1}^k) + \lambda_{j-1}^{k+1} (1 - q_{j-1}^{k+1}) z_{j-1}^{k+1} (S_j^{k+1} - K_{j-1}^k) \right. \\ &\quad \left. \left. + \sum_{i=k+2}^{j-1} \lambda_{j-1}^{i-1} q_{j-1}^{i-1} z_{j-1}^{i-1} (S_j^i - K_{j-1}^k) \right) \right) \\ &\quad + \sum_{i=k+3}^j \lambda_{j-1}^{i-1} z_{j-1}^{i-1} (1 - q_{j-1}^{i-1}) (S_j^{i-1} - K_{j-1}^k) \end{aligned} \quad (\text{A.2})$$

Then we add the first and third term to the summation so that both summation terms run from $i=k+2$ to j , we can combine the summation terms again and expand them:

$$\begin{aligned} e^{r_{j-1}h} * C[K_{j-1}^k, j] &= \lambda_{j-1}^k q_{j-1}^k z_{j-1}^k (S_j^{k+1} - K_{j-1}^k) \\ &\quad + \sum_{i=k+2}^j \lambda_{j-1}^{i-1} q_{j-1}^{i-1} z_{j-1}^{i-1} S_j^i - \lambda_{j-1}^{i-1} q_{j-1}^{i-1} z_{j-1}^{i-1} K_{j-1}^k + \lambda_{j-1}^{i-1} z_{j-1}^{i-1} (1 \\ &\quad - q_{j-1}^{i-1}) S_j^{i-1} - \lambda_{j-1}^{i-1} z_{j-1}^{i-1} (1 - q_{j-1}^{i-1}) K_{j-1}^k \end{aligned} \quad (\text{A.3})$$

so that we use the no-arbitrage condition from equation (2) and use the fact that the probabilities sum to one to arrive at:

$$e^{r_{j-1}h} C[K_{j-1}^k, j] = \lambda_{j-1}^k q_{j-1}^k z_{j-1}^k (S_j^{k+1} - K_{j-1}^k) + \sum_{i=k+2}^j \lambda_{j-1}^{i-1} (F_{j-1, j}^{i-1} - z_{j-1}^{i-1} K_{j-1}^k) \quad (\text{A.4})$$

The two variables that are still not known are q_{j-1}^k and S_j^{k+1} . Therefore we bring these two variables to the left side and set $\rho_{up} = \sum_{i=k+2}^j \lambda_{j-1}^{i-1} z_{j-1}^{i-1} \left(\frac{F_{j-1, j}^{i-1}}{z_{j-1}^{i-1}} - \frac{F_{j-1, j}^k}{z_{j-1}^k} \right)$ as this summation term now consist of forward prices that are all known at $j-1$, use equation (7) and bring $\lambda_{j-1}^{i-1} z_{j-1}^{i-1}$ to the other side:

$$q_{j-1}^k (S_j^{k+1} - K_{j-1}^k) = \frac{e^{r_{j-1}h} C[K_{j-1}^k, j] - \rho_{up}}{\lambda_{j-1}^k z_{j-1}^k} \quad (\text{A.5})$$

Then we can use equation (6), where we change the j counter in equation (6) to $j-1$ and replace q_{j-1}^k in equation (A.5):

$$\left(\frac{K_{j-1}^k - S_j^k}{S_j^{k+1} - S_j^k} \right) (S_j^{k+1} - K_{j-1}^k) = \frac{e^{r_{j-1}h} C[K_{j-1}^k, j] - \rho_{up}}{\lambda_{j-1}^k z_{j-1}^k} \quad (\text{A.6})$$

Now we can solve for S_j^{k+1} :

$$S_j^{k+1} = \frac{S_j^k (C[K_{j-1}^k, j] e^{r_{j-1}h} - \rho_{up}) - \lambda_{j-1}^k z_{j-1}^k K_{j-1}^k (K_{j-1}^k - S_j^k)}{C[K_{j-1}^k, j] e^{r_{j-1}h} - \rho_{up} - \lambda_{j-1}^k z_{j-1}^k (K_{j-1}^k - S_j^k)} \quad (\text{A.11})$$

which is equation (16)

Appendix B Intermediate steps to go from equation (21) to equation (22).

We start with equation (21)

$$S_j^{k+1} = \frac{\frac{(K_{j-1}^k)^2}{S_j^{k+1}} (C[K_{j-1}^k, j] e^{r_{j-1}h} - \rho_{up}) - \lambda_{j-1}^k z_{j-1}^{k,k+1} K_{j-1}^k \left(K_{j-1}^k - \frac{(K_{j-1}^k)^2}{S_j^{k+1}} \right)}{C[K_{j-1}^k, j] e^{r_{j-1}h} - \rho_{up} - \lambda_{j-1}^k z_{j-1}^k \left(K_{j-1}^k - \frac{(K_{j-1}^k)^2}{S_j^{k+1}} \right)} \quad (\text{B.1})$$

First we multiply both sides with the denominator on the right and bring the right side together using the same denominator:

$$S_j^{k+1} \left(C[K_{j-1}^k, j] e^{r_{j-1}h} - \rho_{up} - \lambda_{j-1}^k z_{j-1}^k \left(K_{j-1}^k - \frac{(K_{j-1}^k)^2}{S_j^{k+1}} \right) \right) = \frac{(K_{j-1}^k)^2 (C[K_{j-1}^k, j] e^{r_{j-1}h} - \rho_{up}) - \left(\lambda_{j-1}^k z_{j-1}^k K_{j-1}^k (S_j^{k+1} K_{j-1}^k - (K_{j-1}^k)^2) \right)}{S_j^{k+1}} \quad (\text{B.2})$$

Multiply both sides with S_j^{k+1} , expand the both sides, bring everything to the left side and collect in terms of $(S_j^{k+1})^2$, $\left(\frac{F_{j-1}^k}{z_{j-1}^k}\right)^2$ and S_j^{k+1}

$$(S_j^{k+1})^2 (C[K_{j-1}^k, j] * e^{r_{j-1}h} - \rho_{up} - \lambda_{j-1}^k F_{j-1,j}^k) - (K_{j-1}^k)^2 (C[K_{j-1}^k, j] e^{r_{j-1}h} - \rho_{up} + \lambda_{j-1}^k F_{j-1,j}^k) + 2S_j^{k+1} \lambda_{j-1}^k z_{j-1}^k (K_{j-1}^k)^2 \quad (\text{B.3})$$

Then factoring:

$$-(F_{j-1,j}^k - S_j^{k+1} z_{j-1}^k) \left(C[K_{j-1}^k, j] e^{r_{j-1}h} F_{j-1,j}^k + \lambda_{j-1}^k (F_{j-1,j}^k)^2 - \rho_{up} F_{j-1,j}^k + C[K_{j-1}^k, j] e^{r_{j-1}h} S_j^{k+1} z_{j-1}^k - F_{j-1,j}^k \lambda_{j-1}^k S_j^{k+1} z_{j-1}^k - \rho_{up} S_j^{k+1} z_{j-1}^k \right) = 0 \quad (\text{B.4})$$

We can split the above into two equations

$$-(F_{j-1,j}^k - S_j^{k+1} z_{j-1}^k) = 0 \quad (\text{B.5})$$

and

$$\left(C[K_{j-1}^k, j] e^{r_{j-1}h} F_{j-1,j}^k + \lambda_{j-1}^k (F_{j-1,j}^k)^2 - \rho_{up} F_{j-1,j}^k + C[K_{j-1}^k, j] e^{r_{j-1}h} S_j^{k+1} z_{j-1}^k - F_{j-1,j}^k \lambda_{j-1}^k S_j^{k+1} z_{j-1}^k - \rho_{up} S_j^{k+1} z_{j-1}^k \right) = 0 \quad (\text{B.6})$$

This last equation can be easily solved. Collecting the S and bringing all rest to the right:

$$S_j^{k+1} (C[K_{j-1}^k, j] e^{r_{j-1}h} z_{j-1}^k - F_{j-1,j}^k \lambda_{j-1}^k z_{j-1}^k - \rho_{up} z_{j-1}^k) = -C[K_{j-1}^k, j] e^{r_{j-1}h} F_{j-1,j}^k - \lambda_{j-1}^k (F_{j-1,j}^k)^2 + \rho_{up} F_{j-1,j}^k \quad (\text{B.7})$$

Dividing the right side with everything on the left side except S_j^{k+1} and factoring by K_{j-1}^k :

$$S_j^{k+1} = K_{j-1}^k \frac{\lambda_{j-1}^k F_{j-1,j}^k + (C[K_{j-1}^k, j] e^{r_{j-1}h} - \rho_{up})}{\lambda_{j-1}^k F_{j-1,j}^k - (C[K_{j-1}^k, j] e^{r_{j-1}h} - \rho_{up})} \quad \forall j = \text{odd} \quad \& \quad k = (j+1)/2 \quad (\text{B.8})$$

which is equation (22).

Appendix C Intermediate steps to go from equation (24) to equation (27)

Equation (24) states that the value of a put option is equal to:

$$P(K_{j-1}^k, j) = \sum_{i=0}^j \lambda_j^i \max(K_{j-1}^k - S_j^i, 0) + \lambda_j^{jtd} \max(K_{j-1}^k - 0, 0) \quad (C.1)$$

We know that $S_j^{i+1} > K_{j-1}^k$ and that $S_j^i < K_{j-1}^k$ and since we look at a put option we are only interested in stock prices that are below the strike price and therefore we can reduce equation (C.1):

$$P(K_{j-1}^k, j) = \sum_{i=0}^k \lambda_j^i (K_{j-1}^k - S_j^i) + \lambda_j^{jtd} K_{j-1}^k \quad (C.2)$$

Then we take out $i=0$ and $i=k$ from the summation and replace the lambdas with the ones from equations (8) to (10)

$$P(K_{j-1}^k, j) = (e^{-r_{j-1}h} \lambda_{j-1}^0 z_{j-1}^0 (1 - q_{j-1}^0)) (K_{j-1}^k - S_j^0) + (e^{-r_{j-1}\Delta t} (\lambda_{j-1}^{k-1} z_{j-1}^{k-1} q_{j-1}^{k-1} + \lambda_{j-1}^k z_{j-1}^k (1 - q_{j-1}^k))) (K_{j-1}^k - S_j^k) + \sum_{i=1}^{k-1} (e^{-r_{j-1}h} (\lambda_{j-1}^{i-1} z_{j-1}^{i-1} q_{j-1}^{i-1} + \lambda_{j-1}^i z_{j-1}^i (1 - q_{j-1}^i))) (K_{j-1}^k - S_j^i) + \lambda_j^{jtd} K_{j-1}^k \quad (C.3)$$

Additionally we can express the jump to default Arrow-Debreu price as expressed in equation (26) and replace it in equation (C.3), separate the summation term in two terms and reorder:

$$e^{r_{j-1}h} P[K_{j-1}^k, j] = (\lambda_{j-1}^0 z_{j-1}^0 (1 - q_{j-1}^0)) (K_{j-1}^k - S_j^0) + (\lambda_{j-1}^{k-1} z_{j-1}^{k-1} q_{j-1}^{k-1}) (K_{j-1}^k - S_j^k) + (\lambda_{j-1}^k z_{j-1}^k (1 - q_{j-1}^k)) (K_{j-1}^k - S_j^k) + \sum_{i=1}^{k-1} (\lambda_{j-1}^{i-1} z_{j-1}^{i-1} q_{j-1}^{i-1}) (K_{j-1}^k - S_j^i) + \sum_{i=1}^{k-1} (\lambda_{j-1}^i z_{j-1}^i (1 - q_{j-1}^i)) (K_{j-1}^k - S_j^i) + \left(\lambda_{j-1}^{jtd} + \sum_{i=0}^k \lambda_{j-1}^i (1 - z_{j-1}^i) \right) K_{j-1}^k \quad (C.4)$$

Now we change the counters in the first (summing) term from $\sum_{i=1}^{k-1}$ to $\sum_{i=0}^{k-2}$ and then we can add the first and second term to the summing so that both summing terms run from $i=0$ to $k-1$ and our equation reduces to:

$$e^{r_{j-1}h} P[K_{j-1}^k, j] = (\lambda_{j-1}^k z_{j-1}^k (1 - q_{j-1}^k)) (K_{j-1}^k - S_j^k) + \sum_{i=0}^{k-1} (\lambda_{j-1}^i z_{j-1}^i q_{j-1}^i) (K_{j-1}^k - S_j^{i+1}) + \sum_{i=0}^{k-1} (\lambda_{j-1}^i z_{j-1}^i (1 - q_{j-1}^i)) (K_{j-1}^k - S_j^i) + \left(\lambda_{j-1}^{jtd} + \sum_{i=0}^k \lambda_{j-1}^i (1 - z_{j-1}^i) \right) K_{j-1}^k \quad (C.5)$$

And then we can combine the two summation terms again and expand it:

$$e^{r_{j-1}h} P[K_{j-1}^k, j] = (\lambda_{j-1}^k z_{j-1}^k (1 - q_{j-1}^k)) (K_{j-1}^k - S_j^k) + \sum_{i=0}^{k-1} ((\lambda_{j-1}^i z_{j-1}^i q_{j-1}^i K_{j-1}^k - \lambda_{j-1}^i z_{j-1}^i q_{j-1}^i S_j^{i+1}) + (\lambda_{j-1}^i z_{j-1}^i (1 - q_{j-1}^i) (K_{j-1}^k - S_j^i)) + \left(\lambda_{j-1}^{jtd} + \sum_{i=0}^k \lambda_{j-1}^i (1 - z_{j-1}^i) \right) K_{j-1}^k \quad (C.6)$$

We can use the no-arbitrage condition from equation (2) and sum the exercise prices to arrive at:

$$e^{r_{j-1}h} P[K_{j-1}^k, j] = (\lambda_{j-1}^k z_{j-1}^k (1 - q_{j-1}^k)) (K_{j-1}^k - S_j^k) + \sum_{i=0}^{k-1} (\lambda_{j-1}^i (z_{j-1}^i K_{j-1}^k - F_{j-1,j}^i)) + \left(\lambda_{j-1}^{jtd} + \sum_{i=0}^k \lambda_{j-1}^i (1 - z_{j-1}^i) \right) K_{j-1}^k \quad (C.7)$$

The two variables that are still not known are q_{j-1}^k and S_j^k . Therefore, bringing these two variables to the left side:

$$\begin{aligned}
& (\lambda_{j-1}^k z_{j-1}^k (1 - q_{j-1}^k)) (K_{j-1}^k - S_j^k) \\
& = e^{r_{j-1}h} P[K_{j-1}^k, j] - \sum_{i=0}^{k-1} (\lambda_{j-1}^i (z_{j-1}^i K_{j-1}^k - F_{j-1,j}^i)) - \left(\lambda_{j-1}^{jtd} + \sum_{i=0}^K \lambda_{j-1}^i (1 - z_{j-1}^i) \right) K_{j-1}^k
\end{aligned} \tag{C.8}$$

The term $\sum_{i=0}^{k-1} (\lambda_{j-1}^i (z_{j-1}^i K_{j-1}^k - F_{j-1,j}^i))$ on the right hand side is the sum of all the forwards that lie below the current node and we set that equal to ρ_{down} . The term $\left(\lambda_{j-1}^{jtd} + \sum_{i=0}^K \lambda_{j-1}^i (1 - z_{j-1}^i) \right) K_{j-1}^k$ on the right hand side is all known in period j and if we use equation (25) and (26) we can set that equal to $P_{jtd}[K_{j-1}^k, j] e^{r_{j-1}\Delta t}$

Therefore, when we bring $\lambda_{j-1}^k z_{j-1}^k$ to the other side and replace q_{j-1}^k with the no arbitrage condition equation (13): $q_j^i = \frac{K_{j-1}^k - S_{j+1}^i}{S_{j+1}^{i+1} - S_{j+1}^i}$. This is the same as replacing $1 - q_{j-1}^k$ with $\frac{S_j^{k+1} - K_{j-1}^k}{S_j^{k+1} - S_j^k}$ ³⁷

$$\frac{S_j^{k+1} - K_{j-1}^k}{S_j^{k+1} - S_j^k} (K_{j-1}^k - S_j^k) = \frac{e^{r_{j-1}h} P[K_{j-1}^k, j] - \rho_{down} - P_{jtd}[K_{j-1}^k, j] e^{r_{j-1}h}}{\lambda_{j-1}^k z_{j-1}^k} \tag{C.9}$$

Bring $S_j^{k+1} - K_{j-1}^k$ to the right side, multiply both sides by $S_j^{k+1} - S_j^k$ and subtract K_{j-1}^k from both sides:

$$-S_j^k = \frac{e^{r_{j-1}h} P[K_{j-1}^k, j] - \rho_{down} - P_{jtd}[K_{j-1}^k, j] e^{r_{j-1}h}}{(\lambda_{j-1}^k z_{j-1}^k) (S_j^{k+1} - K_{j-1}^k)} (S_j^{k+1} - S_j^k) - K_{j-1}^k \tag{C.10}$$

Separating both terms on the right side and bring S_j^k to the left, bring it under the same denominator and rearrange, we obtain:

$$S_j^k = \frac{S_j^{k+1} (e^{r_{j-1}h} (P_{df}[K_{j-1}^k, j]) - \rho_{down}) - \lambda_{j-1}^k F_{j-1,j}^k (S_j^{k+1} - K_{j-1}^k)}{e^{r_{j-1}h} (P_{df}[K_{j-1}^k, j]) - \rho_{down} - \lambda_{j-1}^k z_{j-1}^k (S_j^{k+1} - K_{j-1}^k)} \tag{C.11}$$

which is equation (27) with: $\rho_{down} = \sum_{i=0}^{k-1} \lambda_{j-1}^i z_{j-1}^i * \left(K_{j-1}^k - \frac{F_{j-1,j}^i}{z_{j-1}^i} \right)$, $P_{jtd}[K, j] = e^{-r_{j-1}h} \left(\lambda_{j-1}^{jtd} + \sum_{i=0}^K \lambda_{j-1}^i (1 - z_{j-1}^i) \right) K_{j-1}^k$ and $P_{df} K_{j-1}^k, j = P K_{j-1}^k, j - P_{jtd} K_{j-1}^k, j$.

³⁷ We also change the subscripting to $j-1$.