Augmenting the Funded Ratio:
New Metrics for Liability Based Plans

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Abstract

The primary metric for the health of a liability based plan (LBP) is the ratio of the LBP's current assets to its present-valued liabilities. This "funded ratio" cannot address some important financial factors, so we suggest three additional metrics of financial health, connected to the probability of fulfilling the plan's liabilities. The first two metrics compare the current assets and projected future contributions to those needed to attain either (1) a specified probability for meeting *all* the liabilities (SAM, the solvency assets multiple) or (2) specified probabilities for meeting *each* liability (FAM, the funded assets multiple). The third metric, the risk-free funded ratio (RFFR), uses the STRIPS curve to determine the fraction of the liabilities that can be covered without risk. We implement these metrics, first using Monte Carlo simulation given a fixed investment portfolio strategy, and then using dynamic programming to optimize investment portfolio strategies that maximize SAM and FAM.

Highlights and Key Takeaways

- We detail a number of deficiencies that arise if the funded ratio is used as the sole metric of a liability-based plan's health.
- We develop three new metrics (denoted SAM, FAM, and RFFR) that address these deficiencies, augmenting the funded ratio and allowing for a more comprehensive assessment of the health of a liability-based plan.
- These new metrics can be based on static investment strategies or even optimized dynamic investment strategies. We present examples that show how these metrics, in contrast to the funded ratio, explicitly account for a variety of sources of stochasticity, such as in equity returns, interest rates, and inflation, allowing the sensitivity of a plan's health to these and other factors to be determined.

Keywords: funded ratio, liability-based plans, risk metrics
JEL Codes: G11, G40, G41, G51

1 Introduction

We define Liability Based Plans (LBPs) to encompass any plan whose primary investment goal is to fulfill a set of liabilities to be paid over the course of years, like a pension. LBPs include Liability Driven Investing (LDI), Liability Aware Investing (LAI), and Asset-Liability Management (ALM). The distinction between these three categories is not a sharp one but is generally connected to how stochastic the liabilities are assumed to be and how stochastic the assets are allowed to be. LDI is generally viewed as the most conservative of the three. Its liabilities are often assumed to be fixed, and the assets comprise large proportions of bonds for cash-flow matching. ALM is generally the least conservative. ALM often allows for a wide range of asset classes to match liabilities, which are assumed to have varying levels of stochasticity. In typical ALM, the use of equities in the asset portfolio is allowed to be widespread. LAI lies in between LDI and ALM.

While these distinctions between LDI, LAI, and ALM are not completely concrete nor agreed upon, what matters more in the context of this paper is that, regardless of these distinctions, all LBPs employ asset management approaches that are liability-focused, with the primary concern of maintaining a high probability of meeting each liability.[1] This institutional investing priority has interesting parallels to goals-based wealth management (GBWM) for single or multiple goals, in which an optimal investment plan and, in addition for multiple goals, an optimal goals fulfillment strategy is determined for individual investors (as in Das et al. (2020) and Das et al. (2022)). For example, a common previous application of GBWM has been to enable individuals to optimize their defined contribution (DC) plans, but, as we will show in this article, it can also be applied to help pension firms support defined benefit (DB) plans. This comes from the simple recognition that addressing all the LBP's liabilities can be thought of as a goal to be attained from a GBWM context, or, alternatively, addressing each year's liability can be thought of as separate goals to be attained.

While LBPs take varied investment approaches, there is a standard metric commonly used for the health of these plans. This metric is the funded ratio of the plan, which is defined by

$$\text{Funded Ratio} = \frac{\text{Current Assets}}{\text{Current Liabilities}},$$

or some variation of this equation. The "Current Assets" in the numerator is straightforward to calculate. It is the current market worth of the plan's assets. But the "Current Liabilities" in the denominator, also referred to as the projected benefit obligation (PBO), is not as clearly determined, because liabilities are to be paid in the future, and therefore, after the amounts of the future liabilities are estimated, they must be present-valued using a discount rate. The discount rate is a point of much contention. Currently, a fixed discount rate (usually between 6–8%) is chosen to present-value all liabilities, irrespective of their maturities.

[1] https://www.investopedia.com/terms/a/asset-liabilitymanagement.asp

Using the funded ratio as the sole metric of a plan's health potentially fails to take into account a number of important financial issues that we will detail in Section 3. As the American Academy of Actuaries (AAA) Senior Pension Fellow Don Fuerst stated, "Any realistic assessment of a pension plan should include several measures, not just one."[2] The AAA's 2017 issue brief "Assessing Pension Plan Health: More Than One Right Number Tells The Whole Story"[3] is even more clear, stating that:

> "A single number often cannot comprehensively address an issue as complex as the obligation or funded status of a pension plan. The availability of multiple measurements can lead to a more robust understanding of the situation and more well-reasoned conclusions. Understanding that there is more than one right number is an essential step toward engaging in critical issues of retirement security."

This AAA brief then recommends two additional types of measurements, without providing specifics. One type would "represent an estimate of how much money the plan would need to have in order for a projection to show that the assets are expected to be sufficient to cover projected benefit payments," noting that "such an estimate is inherently uncertain." The other type would "rel[y] only on financial information available in today's financial markets" and would be connected "to show[ing] how much it would cost a plan sponsor to transfer the responsibility of supporting a plan to an insurance company or other financial institution."

In this paper, we develop three new metrics. The first two of our metrics, which are distinctive in that they are based on the probability of meeting the plan's liabilities, fit perfectly with the first type of measurement recommended above by the AAA, while our third metric fits perfectly with the second type of AAA-recommended measurement.

- The first metric is the solvency asset multiple (SAM). Here we look for α, the minimum constant multiple of the current assets and future projected contributions (if the plan is open) that is needed to attain a specified probability of meeting all the plan's future liabilities. We then define SAM to be $\frac{1}{\alpha}$, the reciprocal of α. We define SAM by the reciprocal of α, as opposed to α itself, so that it parallels the funded ratio in that a high number is good, and a low number is bad. Also, in both SAM and the funded ratio a value of one means that we are projected to barely meet our goal with the current resources.

- The second metric is the funded asset multiple (FAM). It is defined the same way as SAM, except that instead of looking to attain a specified probability of meeting *all* the plan's future liabilities, we now look to attain specified probabilities for meeting *each* year's future liabilities. This has the advantage of being able to specify higher required probabilities for meeting liabilities in early years

[2] https://tinyurl.com/2ypzfprj
[3] https://tinyurl.com/4ftae3r4

and lower probabilities in later years, where long-term investments like stock may be more appropriate.

- The third metric is the risk-free funded ratio (RFFR). We define RFFR the same way the funded ratio is defined, except that instead of a constant discount rate being used to present-value all the liabilities, we use the yield curve for U.S. Treasury STRIPS to present-value each year's liability separately and we also use this yield curve to present-value future contributions, which we add to the current assets. Because we use STRIPS, RFFR is the ratio of the current plan's worth to the amount needed by the plan to lock in payments of all the future liabilities, assuming those are fixed values. This makes RFFR a highly relevant metric for institutions currently considering buying or selling an LBP. In practice, because future liabilities are often made uncertain by inflation, an LBP's RFFR generally needs to attain a specific value greater than 1, such as 1.3, before the company owning the LBP can immunize themselves by transferring their plan to another institution willing to take it over. We look to compute the cumulative probability over time that RFFR reaches this immunization value, guaranteeing the plan's remaining liabilities will be met.

Because, as the 2017 AAA brief points out, there is inherent uncertainty involved with SAM and FAM, we require models for any stochastic features like inflation or how investments evolve that are used to determine SAM or FAM. These must be determined by the companies managing the LBPs (or regulatory agencies). RFFR, on the other hand, only depends on financial information available in the market, as the AAA recommended, namely the yield curve for STRIPS.

In Section 2, we discuss previous research on metrics for the health of LBPs and methods to optimize these metrics. In Section 3, we will discuss a number of concerns about the funded ratio and how our new metrics can help address these concerns. In Section 4, we show how Monte Carlo simulation can be used to compute our three metrics in the context of a plan where we know the investment strategy and have a variety of sources of uncertainty. We will also discuss the insights that these three metrics provide. In Section 5, we go a step farther, and show how dynamic programming can be used to determine the optimal investment strategy that maximizes SAM or that maximizes FAM. We make some brief concluding comments in Section 6.

2 Previous Research on Measuring and Optimizing Plans with Liability Based Goals

Pension plans are required to report their funded ratio, so that it is publicly known whether the plan is underfunded or overfunded. The present value of liabilities depends on the discount rate applied for discounting. The PPA (Pension Protection Act of 2006)[4] and FASB (US Financial Accounting Standards Board) Statement 158[5] require plan sponsors to fund any shortfalls over certain time horizons and report their liabilities and funding status directly on their balance sheets. The term structures most commonly used for discounting liabilities are the IRS Discount Curve, which is used for the PPA, and the Citigroup Pension Discount Curve, which is used for FASB 158. Both term structures are based on the yields of high-quality (investment grade) US dollar corporate bonds; thus they reflect both interest rate risk and credit spread risk.

The most common metric in addition to the funded ratio that is used in practice is surplus volatility, meaning the volatility of the assets minus the discounted liabilities. Some papers, such as Sharpe and Tint (1990), Ezra (1991), and Delong et al. (2008), have considered methods to maximize the funded ratio minus a constant multiple of surplus volatility. Butt (2012) presents a multidimensional simulation study to analyze the volatility of DB pension funds' funding ratios. Ang et al. (2013) also incorporates downside risk, but through adding a penalty that is proportional to the value of an option that is in the money when the funded ratio drops below one. Cannon and Tonks (2013) undertakes a cross-country comparison of the risks and funded ratios of pension plans, finding that the risk of wage growth and its effect on returns is so substantial that only strategies that are heavily equity loaded have a reasonable chance of remaining funded. Insurance approaches, which are addressed in Broeders and Chen (2013), are likely to reduce the shortfall probability and the expected loss given a shortfall, but they also lower the probability of high positive returns. Boyce and Ippolito (2002) argues that such insurance schemes are too costly.

Previous research on pension fund management strategy suggest that the optimal investment strategy is sensitive to its constraints, for example, van Binsbergen and Brandt (2016) and Martellini and Milhau (2012). Stockton et al. (2008) approaches LDI by looking to meet a plan's liabilities while constraining the volatility of the funded ratio over time. This can be done by exact cash-flow matching (best for frozen funds) or duration matching, or some combination of the two. Exact matching is difficult because employee turnover, inflation, mortality, and salaries can alter the liabilities, so that it is a moving target. While the risk of employee turnover cannot be hedged, at least duration matching can be used to alleviate much of the interest rate risk that is incurred when creating targeted cash-flows to match uncertain liabilities. Upbin et al. (2012) presents a series of benchmark LDI indexes with target durations between six and sixteen years, against which pension fund performance may be benchmarked.

The optimization literature regarding LDI almost exclusively looks to optimize the expected value of a traditional utility function. While static optimization was originally used, dynamic optimization is now far more common (Huang, 2010). Cox et al. (2013) minimizes the expected value of the funding variation, that is, $E\left[\sum_{t=1}^{\infty}\left(\frac{U(t)}{(1+\rho)^t}\right)^2\right]$,

[4] https://www.investopedia.com/articles/retirement/06/ppa2006.asp

[5] https://www.iasplus.com/en/news/2006/September/news3049

where ρ is a discount rate and $U(t)$ stands for the unfunded liabilities[6] in year t. This is done by determining the optimal asset weights and the optimal normal contribution value. Similar objectives are used by Delong et al. (2008). Josa-Fombellida and Rincón-Zapatero (2008) considers the optimal management of an aggregated dynamic pension fund, with multiple classes of workers with stochastic salaries (Chang and Cheng, 2002). Their objective is to minimize the cost of contributions and maximize the utility of the final surplus, measured as the relative fund level with respect to the mean salary, which is a proxy for the replacement rate.[7]

In Josa-Fombellida et al. (2018), when the plan is initially underfunded, the objective is to minimize $E[(X(T))^2]$, the expected square of the negative surplus, X, at the fund's horizon time, T. But when the plan is initially overfunded, the objective is to maximize $E\left[\frac{(X(T))^{1-\xi}}{1-\xi}\right]$, the expected power law utility of the positive surplus, X, at the fund's horizon, where $\xi > 0$ is the risk aversion coefficient. In Mao and Wen (2021) the expected shortfall per unit cumulative wealth is minimized using CVaR. The flip side of this objective would be to maximize the replacement rate, which is a proxy for maximizing the surplus. Devolder et al. (2003) use a variety of utility functions to study the optimal investment strategy of a plan using annuities to target its liabilities in an ALM setting. Battocchio and Menoncin (2004), followed by Ma (2011), examines optimal investment strategies for maximizing the exponential utility of a DB plan's final wealth when interest rates and inflation are stochastic.

Our approach to measuring and optimizing the health of an LBP fund in this paper is different from previous research in that we do not look to optimize traditional utility functions, because those functions do not connect to the core goal of LBP funds, which is to optimize the probability of meeting its liabilities. Our RFFR metric, for example, directly connects to the ability to completely immunize a plan (Biffis and Blake, 2013), meaning there is a 100% chance of meeting its liabilities, or at least to better inform potential buyers of pension plans looking at undergoing a pension risk transfer for de-risking.[8] Our SAM and FAM metrics are directly connected to the probabilities of meeting either all or each of the fund's liabilities, respectively.

The fact that these metrics are about probability is of crucial importance because probability, not traditional utility, is a concept that is intuitive for a general audience. It is key that these metrics describe the health of LBPs in an intuitive way, not only for those running and regulating an LBP, but also for politicians, labor leaders, and, most importantly, the recipients of the liabilities, who all crave a metric for the health of a plan that is easily understood by everyone, regardless of their technical background.

The funded ratio, on the other hand, is not probability-based. While its basic

[6] https://www.thebalance.com/unfunded-liabilities-definition-and-examples-4159564

[7] The replacement rate, also referred to as the income replacement rate, serves as a way to measure the percentage of a worker's current income that a particular pension-based retirement plan can be expected to produce. See: https://www.investopedia.com/terms/r/replacement-rate.asp.

[8] Pension risk transfer is the matching of pension liabilities with the purchase of an external annuity. This is easier to do when liabilities are fixed and not stochastic. Firms with greater pension risk are more likely to de-risk as shown in Li and Kara (2022).

notion of representing the ratio of a plan's assets to its liabilities is intuitive, its ability to truly convey the health of a plan by itself has a number of limitations, as we discuss in the next section.

3 Limitations of the Funded Ratio; Advantages of the New Metrics

Concerns about using the funded ratio as the sole metric of an LBP's health have arisen from a variety of sources. Sgouros (2017) lists a number of these concerns in the context of funding public pensions, and he presents recommendations for pension accounting standards reform in Sgouros (2019). Concerns about solely using the funded ratio have also been discussed by the AAA, leading to their publishing two issue briefs[9] regarding what they refer to as the "80% funding myth," by which they mean that "[f]requent unchallenged references to [an] 80% fund[ed ratio] as a healthy level threaten to create a mythic standard."

Determining the appropriate discount rate for the funded ratio is a particularly difficult task. Indeed, in some sense it is impossible, not only because it is one constant required to accurately reflect a large variety of financial factors, but also because even the best efforts to approximate the discount rate cannot address other fundamental limitations that the funded ratio has by its nature. We consider nine limitations below, comparing them with SAM, FAM, and RFFR.

1. Inaccurate discount rates only affect liabilities, with no balancing effect on the assets in the funded ratio calculation.

Sgouros (2019) states that metrics like the funded ratio should "not combine low-accuracy numbers (liability) with high-accuracy numbers (assets) and expect to get anything but a very rough estimate, unsuitable as a basis for making important decisions." This imbalance creates an incentive to project high return rates since these decrease the funded ratio's denominator due to the higher discount rate, but have no effect on the numerator. That is, projecting higher returns increases the funded ratio, thereby giving the impression of a healthier plan.

SAM and FAM are not ratios, so they are not vulnerable in the same way. High return rates will also increase SAM and FAM, but a company managing a liability based plan (or a regulatory agency) need only specify clear market parameters such as stock returns, interest rates, inflation, etc. to prevent this. In contrast, the appropriate discount rate is different for each LBP and more complicated to determine, based on each plan's unique financial circumstances. RFFR, on the other hand, is completely free from such problems, since it strictly uses the current yield curve for STRIPS. That is, because it only uses market data, RFFR does not require projecting parameters regarding future stochastic behavior like SAM and FAM do, nor a com-

[9] `https://tinyurl.com/yry2562n` and, more recently, `https://tinyurl.com/4b4z6tzt`

plicated, individualized procedure to determine the most appropriate discount rate like the funded ratio does.

2. Using a discount rate only reflects the expected return, not the volatility, of an LDI fund.

The discount rate is, by definition, a projected *expected return* on the plan's overall assets over the horizons of the liabilities. Even if it is completely accurate for some reason, it cannot take into account the volatility of the returns. When used as the sole metric, this has important, negative ramifications for LDI plans. Since riskier investment portfolios have a higher expected return, they create a higher funded ratio, because the liabilities are discounted at the higher expected rate of return. That is, the more risky the investment portfolio, the healthier the plan will seem to be according to the funded ratio metric, even though LDI plans are supposed to be particularly careful to assess the dangers that such additional volatility creates. Of course, this is exactly why plans look instead to optimize the funded ratio minus a constant multiple of an additional metric like surplus volatility, although this creates the new question of what the value of this constant multiple should be.

SAM and FAM are far superior in this regard since they require no additional metrics, and they can work with the full probability distribution created by as many stochastic factors as are appropriate, not just expected returns and volatilities. RFFR is also superior since it is a metric for what can be guaranteed, meaning there is no volatility to take into account.

3. Liabilities, but not assets, have additional sources of uncertainty that are approximated by constant values.

Aside from the problem of approximating returns for the discount rate, liabilities suffer from other sources of uncertainly, such as inflation rates and mortality rates, which change the size of the liabilities. However, because the denominator in the funded ratio needs be a specific number, estimated liability amounts are used for each year, which are then present-valued. This again means that the liability computation fails to account for stochastic behavior, while the assets computation remains deterministic.

In contrast, SAM and FAM are allowed to work with as many stochastic factors as desired, including those for the sizes of the liabilities. RFFR, like the funded ratio, must use each year's projected, estimated liability.

4. The funded ratio doesn't reflect the importance of the liabilities' due dates.

The funded ratio does not reveal anything about the timing of the liabilities, except through the discount rate. But a funded ratio of 80% is not remotely as serious a concern if the liabilities are due 30 years from now, as opposed to if the liabilities are due 1 year from now.

SAM and FAM both take into account the time when each liability occurs. Both metrics would be strongly affected by liabilities being due 30 years from now versus 1 year from now. Because RFFR pertains solely to risk-free investing, it makes little difference if liabilities are due 30 years from now or 1 year from now.

5. Using one discount rate for all time horizons is problematic.

Because there is only one discount rate being considered, it is not possible to consider applying a different rate to short-term liabilities versus long-term liabilities. The rate applied to each year's liabilities must be the same. This ignores the shape of the yield curve, and it ignores the differences in short-term versus long-term asset holdings (that is, the glide path of the portfolio).

SAM and FAM do not need or use discount rates, and further, the market parameters used in SAM or FAM do not need to be constants. The parameters can be deterministic or stochastic functions of time. RFFR, of course, applies different rates to short-term liabilities versus long-term liabilities, depending on the yield curve for STRIPS.

6. Future contributions and new liabilities are not addressed in the funded ratio.

This means that a plan that is about to become closed (meaning closed to future contributions and the corresponding new liabilities) is viewed as being as healthy as an identical plan that stays open (meaning open to future contributions and the corresponding new liabilities), even though the open plan is far more likely to remain solvent in the short term.

In contrast, SAM, FAM, and RFFR take into account all projected future contributions and can also take into account all projected future liabilities. With that said, RFFR is a metric that is mostly relevant to closed plans. That is, RFFR is concerned with valuing the current worth of transferring a plan, but transferring an open plan makes less sense since the new owner would be agreeing to pay new liabilities that would continue to be created by the old owner.

7. Augmenting the funded ratio to address future contributions and new liabilities leads to new problems.

A seemingly straightforward solution to the problem discussed in the previous point is to alter the funded ratio to take projections for future contributions and new liabilities into account, using the discount factor to present-value estimates of both future contributions and new liabilities. We will refer to this as the "augmented funded ratio" in the next section. The augmented funded ratio, however, has its own problems: (1) First, the augmented funded ratio does not account for the need to have sufficient contributions arrive before liabilities are due. For example, let's say a contribution will occur 20 years from now, and it is large enough to offset a liability due 21 years from now. But if the contribution were to occur 22 years from now, instead of 20, it could not offset the liability due in year 21 and a default would occur. From the perspective of the augmented funded ratio, however, both of these contributions would simply just be present-valued, leading to nearly the same measure of portfolio health. (2) Second, the projected contributions and liabilities that are equal in size and occur on the same day in the future should just cancel each other and have no influence on the measurement of the plan's health. Instead, their present values will be added to the numerator and the denominator in the augmented funded ratio, meaning that the larger they are, the more they artificially move the augmented funded ratio towards one, distorting the measure of the actual health of the plan. A potential cure for this issue is to simply cancel projected contributions and equally large liabilities that are projected to occur at the same time. However, defining what "the same time" means creates new issues. It could be defined to mean "within the same month" or "within the same year," etc. No matter the definition, being just outside this time frame would then trigger present-valuing the projected contributions and liabilities, instead of canceling them, leading to a suddenly different augmented funded ratio than if they were just inside the time frame.

SAM and FAM have none of these problems. They account for the need to have contributions arrive before liabilities are due. Projected future contributions and liabilities that are equal in size, no matter how big that size is, will simply cancel if they occur at the same time. Better, as the time difference between a big liability and an equally big future contribution gets larger, SAM and FAM adjust to the larger difference in a continuous way. While RFFR can suffer from the same problems just described for the augmented funded ratio, it is unlikely to since RFFR is primarily relevant to closed plans where there is no need to accommodate future contributions and corresponding new liabilities.

8. The funded ratio does not directly take into account an employers' financial health.

While a liability based fund running out of money is never desirable, it is less serious if it is backed by a large governmental body that is able to supplement the plan with additional money, and more serious if it is backed by a small private employer with less access to additional resources. Current methods to address this important difference are somewhat artificial. For example, for a large governmental body, the discount rate may be allowed to represent the expected return of a 60% stock/40% bond fund, whereas, for a small, private body, it becomes based on, say, the BBB corporate bond return rates.

For SAM and FAM, this can be accommodated in a direct manner. Specifically, the more additional resources the employer has access to, the lower the required probability of remaining solvent (for SAM) can be and the lower the required probabilities for paying each year's liability (for FAM) can be. While this is akin to allowing a lower or higher funded ratio, it has the distinct advantage of having a directly understandable probabilistic meaning, in contrast to the funded ratio, which has a less direct interpretation. Since RFFR only pertains to the risk-free case, concerns about the financial health of the employer are not relevant.

9. The funded ratio provides no useful information about how a liability based plan should be invested.

The funded ratio is a measurement that is unaffected by how an LBP is invested, except through the discount rate. As noted before, this means the only investment advice the funded ratio suggests is to adopt the most aggressive investment portfolio available since that maximizes the discount rate. Of course, this is poor advice for most liability based plans.

SAM and FAM, in contrast, are very useful metrics for optimizing an investment plan. In Section 4, just below, we will see how to calculate these metrics for a fixed investment strategy. This enables comparing any number of fixed investment strategies and then selecting the strategy with the highest SAM (or highest FAM) value. In Section 5, we will go much farther, showing how to determine the dynamic investment strategy over time that maximizes the value of SAM (or FAM), meaning we calculate how to best meet the obligations of the liability based plan. RFFR assumes risk-free investing in STRIPS, so the investment portfolio is assumed not to vary from STRIPS.

4 Computing and Working the New Metrics Using Monte Carlo Simulation for a Fixed Investment Strategy

In this section, we consider funds with a fixed portfolio strategy. For example, an LDI fund might have a fixed strategy of 90% bonds and 10% stock. Target date funds are another example of a fixed strategy, because even though the fractions of their underlying components change over time, they do so in a specified (fixed) way.

Because the portfolio strategy is fixed, Monte Carlo simulation is an effective tool to gauge the effect of a variety of sources of randomness that can affect the performance of the fund and its ability to address liabilities. In our examples in this section, we will consider three sources of randomness: movement in the stock market, changes to the inflation rate, and changes to the yield curve for interest rates. We note that we could also easily include the effects of varying mortality rates or uncertainty in contribution rates, bond defaults, etc. Our model will only consider investing in a total stock market index fund and AA rated corporate bonds. Again, we note that we could easily expand our model to include the effect of additional assets like commodities, derivative securities, international stocks, high-yield bonds, TIPS, Treasury bonds, etc. To calculate RFFR, our model will also need to use STRIPS.

For our fixed portfolio strategy, we look to use Monte Carlo simulation to calculate SAM or FAM or to determine the likelihood of being able to immunize a plan if it attains a sufficiently high RFFR and can therefore be sold. To do this, we must describe the nature of the plan and our assumptions for modeling both our sources of uncertainty and for the fixed portfolio strategy. We do this first for SAM and FAM. We will later discuss the additional modeling necessary for the STRIPS needed to determine RFFR.

4.1 The nature of the plan and calculating the funded ratio

We consider a plan that currently has \$80MM in assets. Over the next five years, there will be contributions of \$5MM, \$4MM, \$3MM, \$2MM, and \$1MM nominal dollars and then the plan will close, so we have no further contributions after five years. This corresponds to α, the asset and contribution multiplier, equaling one. If, say, we had $\alpha = 1.1$, we would start with \$88MM in assets and then have contributions over the first five years of \$5.5MM, \$4.4MM, \$3.3MM, \$2.2MM, and \$1.1MM nominal dollars.

We assume the plan's liabilities require it to pay \$5MM present-valued dollars in each of the next 15 years, adjusted by inflation. In each of the 15 years after that, we assume the plan's liabilities require it to pay \$4MM present-valued dollars, adjusted by inflation. We assume no taxes on the portfolio's bonds or stocks.

The information above (for $\alpha = 1$), along with a discount rate, D, and an estimate for the inflation rate, $i_{\text{infl,estim}}$, which must both be selected, gives us sufficient information to calculate the funded ratio:

$$\text{Funded ratio} = \frac{\text{Current Assets}}{\text{Current Liabilities}} = \frac{80}{\sum_{t=1}^{15} 5e^{(i_{\text{infl,estim}}-D)t} + \sum_{t=16}^{30} 4e^{(i_{\text{infl,estim}}-D)t}}. \tag{1}$$

We can also, if desired, use the augmented funded ratio defined in the previous section, which allows the contributions arriving in the first five years to be included as part of the assets:

$$\text{Augmented funded ratio} = \frac{80 + \sum_{t=1}^{5}(6-t)e^{-Dt}}{\sum_{t=1}^{15} 5e^{(i_{\text{infl,estim}}-D)t} + \sum_{t=16}^{30} 4e^{(i_{\text{infl,estim}}-D)t}}. \tag{2}$$

Note that because the contributions arrive well in advance of most of the liabilities, the concerns outlined in the previous section for the augmented funded ratio do not materialize in the case we are considering here. Unless otherwise specified, we will use $i_{\text{infl,estim}} = 3\%$. In general, D should take a value between 3.5% and 7% to reflect the fact that we will later assume that θ_{AA}, the average instantaneous rate of return of the AA corporate bond position, is 3.5%, and that μ, the average return on the stock position, is 7%. Table 1 presents both the funded ratio and the augmented funded ratio for values of D in a wider range since we will eventually look at cases where we alter θ_{AA} and μ in this wider range.

Using the augmented funded ratio makes more sense than the normal funded ratio, especially given how early the contributions occur. We note from Table 1 that the augmented funded ratio is greater than 0.8 — the ratio commonly associated with a healthy fund — for all but one of the D values in the 3.5% to 7% range. We will look to more carefully explore how healthy this fund actually is in an LDI context once we have specified our full model and can run our Monte Carlo simulations.

Discount rate (D)	Funded ratio	Augmented funded ratio
1.0%	0.435	0.515
1.5%	0.472	0.557
2.0%	0.510	0.601
2.5%	0.550	0.648
3.0%	0.593	0.696
3.5%	0.637	0.747
4.0%	0.684	0.801
4.5%	0.732	0.856
5.0%	0.783	0.914
5.5%	0.836	0.974
6.0%	0.890	1.036
6.5%	0.947	1.100
7.0%	1.006	1.166
7.5%	1.066	1.235
8.0%	1.128	1.305
8.5%	1.192	1.377
9.0%	1.258	1.450
9.5%	1.325	1.526
10.0%	1.394	1.603

Table 1: The computed funded ratio given in equation (1) and the computed augmented funded ratio given in equation (2) for a variety of discount rates when $i_{\text{infl,estim}}$, the inflation estimate, is 3%. The funded ratio does not include projected future contributions, whereas the augmented funded ratio does include them.

4.2 Sources of randomness in the model

We consider three sources of randomness:

- Stock: For simplicity, we assume the stock component's price evolves by geometric Brownian motion, which means that $S(t)$, the stock component's price at time t, is governed by the stochastic differential equation

$$S(t+1) = S(t)e^{\mu - \frac{\sigma^2}{2} + \sigma Z_1},$$

  where Z_1 is a standard normal random variable. We have chosen $\mu = 7\%$ and $\sigma = 20\%$ as an example of a forecast for these parameters if the stock represents a total stock index.

- Inflation: We assume that inflation evolves by the Vasicek (1977) model, which means that $i_{\text{infl}}(t)$, the rate of inflation at time t, is governed by the stochastic differential equation

$$di_{\text{infl}} = \kappa_{\text{infl}}(\theta_{\text{infl}} - i_{\text{infl}})dt + \sigma_{\text{infl}} dW, \tag{3}$$

where W is a Wiener process. The Vasicek model is an Ornstein–Uhlenbeck process (Uhlenbeck and Ornstein, 1930) for mean reversion, where θ is the mean value (of i_{infl} in this case), κ is the strength of the reversion to this mean value, and σ is a constant representing the strength of the randomness. The solution to equation (3) is

$$i_{\text{infl}}(t+1) = i_{\text{infl}}(t)e^{-\kappa_{\text{infl}}} + \theta_{\text{infl}}(1 - e^{-\kappa_{\text{infl}}}) + \sigma_{\text{infl}} \sqrt{\frac{1 - e^{-2\kappa_{\text{infl}}}}{2\kappa_{\text{infl}}}} Z_2, \tag{4}$$

where Z_2 is a standard normal random variable that is independent of Z_1. Zhang and Ewald (2010) derive an optimal investment strategy for a pension fund facing inflation. As an example of a forecast for the parameters in our model, we have chosen $\kappa_{\text{infl}} = 0.6, \theta_{\text{infl}} = 0.025$, and $\sigma_{\text{infl}} = 0.03$, loosely based on United States inflation rate data between 1955 and 2023. We have also chosen an initial inflation rate of 2%.

- The yield curve for zero-coupon AA corporate bonds: We assume that $r_{\text{AA}}(t)$, the instantaneous interest rate of AA corporate bonds at time t, is also governed by the Vasicek model, but with the restriction that it cannot become negative, therefore, from equation (4), we have that

$$r_{\text{AA}}(t+1) = \max\left\{0, r_{\text{AA}}(t)e^{-\kappa_{\text{AA}}} + \theta_{\text{AA}}(1 - e^{-\kappa_{\text{AA}}}) + \sigma_{\text{AA}} \sqrt{\frac{1 - e^{-2\kappa_{\text{AA}}}}{2\kappa_{\text{AA}}}} Z_3\right\}, \tag{5}$$

  where Z_3 is a standard normal random variable that is independent of Z_1 and Z_2. Further, as shown in Mamon (2004) for example, this implies that $ZCB_{\text{AA}}(t, T)$, the cost of a zero-coupon AA corporate bond at time t that will be worth \$1 at time T, is given by

$$ZCB_{\text{AA}}(t, T) = e^{B(t,T) - r_{\text{AA}}(t)A(t,T)}, \tag{6}$$

  where

$$A(t, T) = \frac{1}{\kappa_{\text{AA}}}\left(1 - e^{-\kappa_{\text{AA}}(T-t)}\right)$$

  and

$$B(t, T) = \left(\theta_{\text{AA}} - \frac{\sigma_{\text{AA}}^2}{2\kappa_{\text{AA}}^2}\right)\left(A(t, T) - T + t\right) - \frac{\sigma_{\text{AA}}^2}{4\kappa_{\text{AA}}} A^2(t, T).$$

  As an example of a forecast for these Vasicek model parameters for AA corporate bonds, we have chosen $\kappa_{\text{AA}} = 0.5, \theta_{\text{AA}} = 0.035$, and $\sigma_{\text{AA}} = 0.02$, loosely based on AA corporate yield data between 1996 and 2023. We have also chosen an initial instantaneous interest rate for our AA corporate bonds of 4%.

While we have chosen to use the geometric Brownian motion and Vasicek models above, different models can be substituted to describe these sources of randomness if desired.

4.3 The fixed investment strategy for our model

With our stochastic model in place, we describe a specific fixed investment strategy that we will use throughout this section, noting that, of course, this is just one example of the multitude of possible fixed investment strategies that can be applied.

We assume $i_{\text{infl,estim}}$ is the inflation rate for any years not yet simulated using the Vasicek model. Given this assumption, we take our initial investment and first prioritize purchasing bonds that cover the next five years of liabilities (meaning first covering the liabilities in year 1, then year 2, etc., up to year 5). If there is more money after that, we devote a fraction, f, of that leftover money to purchasing stocks. The remainder of the money is then used to purchase bonds to cover the liabilities in chronological order (i.e., year 6, then year 7, etc.), until we run out of money or we purchase bonds that cover every one of the 30 years, in which case we put the remaining initial investment into stocks.

In each year after that, we apply the following approach:

1. As specified above, we generate the effect of our sources of randomness over the year, which, of course, will be different for each simulated Monte Carlo path. More specifically, 1) we use geometric Brownian motion to determine the change in the worth of the stock position over the course of the year; 2) we use the inflation rate generated by the Vasicek model for the year to adjust both the present and future liability amounts; and 3) we use the generated change to the yield curve for the Vasicek model to update the prices of all old bonds owned and all new bonds of any desired maturity that might be purchased, along with the coupon structure of these new bonds.

2. At the end of the year, we collect any money generated from contributions, bond coupons, and principal from matured bonds to address that year's liabilities. This money may or may not be enough for this purpose.

   (a) If there is enough collected money to pay the year's liability, we apply the same process to the money that remains after paying the liability that we did with the initial money, namely: We first use this money to make sure that bonds have been purchased to cover the next five years of liabilities, again assuming the inflation rate $i_{\text{infl,estim}}$ in future years. If there is more money after that, we devote a fraction, f, of that leftover money to purchasing stocks. The remainder is used to purchase bonds covering the liabilities six years later, then seven years later, etc., until we either run out of money or we purchase bonds to cover the liabilities in year 30, the final year. If we manage to cover year 30, we use the remainder to purchase stock since we have bonds covering our estimates of every year's liability.

   (b) If the collected money is not enough to cover the year's liability, we must sell bonds and stock to pay for the remaining liability. We first sell the bonds with the longest time until maturity, followed by selling a fraction of the stock that is equal to the reciprocal of this longest time until maturity. We repeat this process as necessary, noting that with each repetition, the longest time until maturity is reduced by a year, until we have addressed the remaining liability or we reach maturities that are five years out. In this later case, we sell all the remaining stock needed to address the full liability. Should there not be enough stock to address the liability after we sell all the stock, we sell the bonds with the longest maturities until we have addressed the full liability. Should we not be able to address the full liability after selling all the stock and all the bonds, we are bankrupt.

3. Finally, if the next five years of liabilities are not covered by bonds at this point, we sell as much stock as is necessary to purchase bonds to attain this priority, if possible.

4.4 Base case results when $\alpha = 1$

We are able to run the above investment strategy over 10,000 simulations in about 15 seconds on a home computer with an Apple M1 Max chip. During the run, we determine the fraction of the simulations that go bankrupt each year, which enables us to record the increase in the cumulative fraction of simulations that go bankrupt over time. We use the following values for the parameters discussed above in our base case:

1. f, the fraction of money devoted to stocks after ensuring the next five years of liabilities are covered by bonds, is set to 10% to model LDI plans, which rarely exceed a 10% stock fraction,

2. $\alpha = 1$ for the initial assets and later contributions,

3. the "years of bonds," meaning the number of years we prioritize having bonds cover the projected liabilities, is 5 years,

4. $i_{\text{infl,estim}}$, the estimate of future inflation rate needed to project future liabilities, is 3%,

5. μ and σ, the mean and volatility of the stock governed by geometric Brownian motion, are 7% and 20% respectively,

6. The Vasicek model parameters governing the evolution of inflation are $\kappa_{\text{infl}} = 0.6, \theta_{\text{infl}} = 0.025$, and $\sigma_{\text{infl}} = 0.03$, and

7. The Vasicek model parameters governing the evolution of the yield curve for AA corporate bonds are $\kappa_{\text{AA}} = 0.5, \theta_{\text{AA}} = 0.035$, and $\sigma_{\text{AA}} = 0.02$.

For this base case, we find that there is an 81.9% chance of becoming bankrupt by the end of 30 years. This is detailed in Figure 1. We recall from Table 1 that the (augmented) funded ratio is generally above 0.80 for this collection of parameters, a region generally considered to be safe, but Figure 1 shows that the chance that this LDI fund becomes bankrupt by the end of its 30 years horizon is 81.9%, which is far from what is generally considered to be safe.

This strongly supports the AAA's contention at the beginning of the previous section that using an 80% or higher funded ratio as an indicator of a healthy LBP is a "myth." It also gives further evidence to the AAA's view that the funded ratio should be augmented by additional metrics to give a full view of the health of an LBP. We explore using the SAM and FAM metrics for our base case in the next subsection.

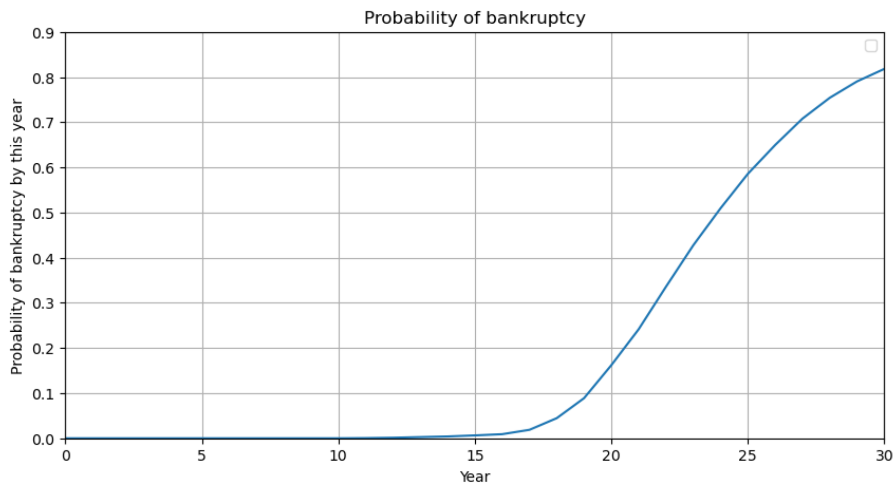

Figure 1: Our base case: The graph shows the cumulative probability over time that the plan will become bankrupt during the 30 years it is tasked with addressing liabilities. The chance of bankruptcy is 81.9% at the end of the 30 years, even though the plan's funded ratio is near or above 0.80, which would generally be deemed safe. (The chance of bankruptcy is 0.70% after 15 years, 16.2% after 20 years, and 57.4% after 25 years.)

4.5 SAM and FAM for the base case

To determine SAM in this section's Monte Carlo context, we determine the minimum multiple, α, of the initial investment and contributions needed to keep the fraction of bankruptcies for our base case below a specified value. SAM is the reciprocal of this minimum α value.

For instance, let's say that we require that we remain under a 20% chance of bankruptcy at the end of the 30 year horizon for SAM. For our base case, we find that SAM $= 0.743$, which corresponds to an α of $1.346 = \frac{1}{\text{SAM}}$. That is, we need to multiply our initial assets and our contributions by 1.346 to bring the chance of bankruptcy down to 20%. When this occurs, we have a 0.03% chance of bankruptcy after 15 years, a 0.28% chance after 20 years, and a 3.64% chance after 25 years.

The definition of FAM given in the introduction looks to keep the probability of failing to meet each annual liability below given values for each year. This definition makes sense in a general LBP context. In this general context, a large fraction of the plan can be in volatile assets, and we do not necessarily want to bias the plan's investment decisions towards meeting early liabilities at the expense of later liabilities. In our LDI context here, however, that bias is often desirable, and further in our Monte Carlo context here, investment decisions are already determined and are heavily weighted towards AA corporate bonds. We therefore utilize a version of FAM in this section that is slightly different, and more helpful in answering questions that arise in an LDI context. Specifically, we determine the minimum multiple, α, of the initial investment and contributions that is needed to keep the *cumulative* fraction of bankruptcies for our base case below specified values for each year. FAM is then defined in this section as the reciprocal of this minimum α value.

In Section 5, we will revert back to the normal definition of FAM. Should the normal definition of FAM be more desirable in the Monte Carlo context here, it can be accomplished by using the best off funds in the Monte Carlo simulation in any given year to address that year's liability. For example, if there is a 1% allowed probability of failing to meet the liability in a given year, the top 99% well-off simulated paths would be used to address the liability. If this is not possible, then a higher α value must be selected.

For this section's version of FAM, let's say that we require that we remain under a 0.5% chance of bankruptcy at the end of 20 years, under a 1.5% chance of bankruptcy at the end of 25 years, and under a 20% chance of bankruptcy at the end of the 30 year horizon. (We note that we can compute FAM just as easily if we decided to specify different chances for bankruptcy in each of the 30 years. Because we are working with cumulative bankruptcies, these specified chances never decrease over time, of course.) From our SAM results, we can see that the key obstacle for FAM will be attaining the 1.5% chance of bankruptcy at the end of year 25, which will require a higher α than 1.346. And indeed, for our base case, FAM = 0.704, which corresponds to an α of $1.421 = \frac{1}{\text{FAM}}$. When this occurs, we have a 0.01% chance of bankruptcy after 15 years, a 0.17% chance after 20 years, a 1.50% chance after 25 years (as expected), and a 12.1% chance after 30 years.

4.6 Comparative statics analysis of the base case

In this section we note the effects of changing each of the base case parameters, one at a time. Only changes to the parameter $i_{\text{infl,estim}}$ directly affect the funded ratio (or the augmented funded ratio), although changing the other parameters can indirectly alter the discount rate D used to determine the funded (and augmented) funded ratios. With SAM and FAM, however, the effect of changing any of these parameters can directly be quantified, as we show in this subsection. This is important because it enables managers of liability based portfolios to directly determine how important it is to hedge for the effects of uncertainty or changes in each of these parameters.

In Table 2, we see the effect of changing f. Recall that f is the fraction of the plan devoted to stock after we have covered the projected liabilities for the next 5 years (or however many years are assigned to what we will call the "years of bonds" parameter). Looking at the cases where $\alpha = 1$ in Table 2 suggests, at first, that plans would be far better off choosing a value of f near 70% instead of the base case value of 10% since $f = 0.7$ minimizes the probability of being bankrupt after 30 years. But it is important to recall that when $\alpha = 1$, there is a high chance of going bankrupt, so it is no surprise that putting such a large fraction of the plan in stocks reduces the probability of bankruptcy: it is essentially a financial "Hail Mary" pass for a plan that requires early intervention despite the high value of its augmented funded ratio. Adjusting α to 1.346, where we know from our SAM calculation that the probability of bankruptcy is a more reasonable value of 20% shows that for this reasonably well-funded plan, keeping f somewhere in the range of 7.5-10% is optimal. That is, increasing the stock fraction above 10% is neither necessary nor wise if the LDI plan is reasonably well funded. Recalling that we want SAM and FAM to be as large as possible, the bottom panel in Table 2 for SAM and FAM confirms that the optimal value of f is in the 7.5-10% range.

		Probability of bankruptcy after...			
α	f	15 years	20 years	25 years	30 years
1	0.05	1.67%	16.8%	61.9%	87.0%
1	0.1	0.70%	16.2%	57.5%	81.9%
1	0.2	0.31%	17.0%	53.5%	74.6%
1	0.3	0.77%	19.7%	51.2%	70.5%
1	0.5	2.59%	23.8%	49.7%	66.1%
1	0.7	5.31%	26.9%	49.7%	64.6%
1	0.9	6.07%	28.2%	51.3%	65.7%
1.346	0.05	0.15%	0.67%	3.38%	20.9%
1.346	0.075	0.09%	0.42%	3.39%	20.0%
1.346	0.1	0.03%	0.28%	3.63%	20.0%
1.346	0.2	0%	0.13%	5.27%	21.5%
1.346	0.3	0%	0.47%	7.84%	23.9%

f	0.05	0.075	0.1	0.2	0.3
SAM	0.738	0.743	0.743	0.735	0.723
FAM	0.702	0.706	0.704	0.682	0.641

Table 2: The effect of changing f, the maximum fraction of the portfolio that can be in stock, unless every projected liability is covered by bonds. Top panel: The effect of changing f on the probability of bankruptcy when we set the asset and contribution multiplier, α, equal to 1 and then when we set α equal to 1.346. Bottom panel: The effect of changing f on SAM and FAM.

The remaining tables in this subsection take the form of the bottom panel in Table 2. That is, we increase a single parameter of interest over the five columns in the table, letting the parameter take its base case value in the middle column. Because both SAM and FAM require the cumulative probability of bankruptcy to be no more than 20% after 30 years, but FAM also requires no more than a 0.5% chance of bankruptcy after 20 years and no more than a 1.5% chance of bankruptcy after 25 years, FAM will never be greater than SAM. Generally, the 20 year requirement for FAM has a larger effect on reducing FAM than the 25 year requirement, except in the base case, where the 25 year requirement has the larger effect.

For example, in the left panel of Table 3, we see the effect of changing the "years of bonds", that is, the number of future years for which bonds must cover projected liabilities before we devote any money to stock. The SAM row in this left panel shows that moving the "years of bonds" within the range of one to five years makes little to no difference to the probability of eventual bankruptcy, but having 10 or more "years of bonds" begins to under-weight stock in favor of potentially very volatile long-term bonds, which does begin to slightly diminish the probability of remaining solvent. The FAM row, on the other hand, shows that while having a single "year of bonds" is a little less cautious than it should be, having 10 or more "years of bonds" is much more problematic, leading to potentially having to sell long-term bonds prematurely, which makes it more difficult to keep the rate of solvency higher than 99.5% after 20 years.

Similarly, in the right panel of Table 3, we see the effect of changing $i_{\text{infl,estim}}$, the inflation estimate used to project future liabilities that are to be covered by bonds. Since the average inflation rate is given by θ_{infl}, which is 2.5%, we see in the panel that when $i_{\text{infl,estim}}$ is higher than 2.5%, there is little effect on SAM or FAM, although both begin to suffer when $i_{\text{infl,estim}}$ is too high, due to losing the benefits of having more stock in the portfolio. When $i_{\text{infl,estim}}$ is too low, especially when it is unrealistically set equal to 0%, we see SAM, and especially FAM, suffer. This is no surprise with FAM since such low values of $i_{\text{infl,estim}}$ lead to far more stock in the portfolio, which creates enough volatility in the plan that maintaining a chance of bankruptcy of no more than 0.5% after 20 years becomes quite difficult.

Years of bonds	1	3	5	10	30
SAM	0.744	0.744	0.743	0.737	0.720
FAM	0.699	0.702	0.704	0.687	0.569

$i_{\text{infl,estim}}$	0%	2%	3%	5%	10%
SAM	0.705	0.743	0.743	0.742	0.741
FAM	0.500	0.612	0.704	0.708	0.703

Table 3: Left panel: the effect of changing the "Years of Bonds," which is the number of future years of projected liabilities that must be covered by bonds before devoting any plan money to stock. Right panel: the effect of changing the inflation estimate, $i_{\text{infl,estim}}$, used to project future liabilities.

In Table 4, we see the beneficial effect of increasing the expected return, μ, of the stock and the detrimental effect of increasing the volatility, σ, of the stock. We note their real, but muted, effects on SAM and FAM due to the limitations on the amount of stock in the plan.

μ	4%	6%	7%	8%	10%
SAM	0.705	0.729	0.743	0.759	0.798
FAM	0.680	0.693	0.704	0.713	0.738

σ	5%	15%	20%	25%	35%
SAM	0.786	0.760	0.743	0.727	0.701
FAM	0.756	0.719	0.704	0.687	0.651

Table 4: The effect of changing the stock's expected value, μ, (left panel) and volatility, σ, (right panel).

In Table 5, we see the effect on SAM and FAM of the three Vasicek model parameters for the rate of inflation. The higher κ_{infl} is, the more push there is towards the mean of $\theta_{\text{infl}} = 2.5\%$. When κ_{infl} is quite small, using the inflation estimate $i_{\text{infl,estim}} = 3\%$ becomes progressively questionable, and so we see SAM decrease significantly, and FAM decrease even more since maintaining a 99.5% solvency probability after 20 years becomes nearly impossible. Even worse, when θ_{infl} increases, we are nearly guaranteeing that the inflation estimate $i_{\text{infl,estim}} = 3\%$ will be wrong, and the liabilities will be far higher than projected when we purchased the bonds. It's no surprise in this case that both SAM and FAM become very low. Increasing the effect of the randomness in the inflation rate by increasing σ_{infl} is similar to the effect of decreasing κ_{infl}, which is borne out by comparing the left and bottom panels in Table 5.

κ_{infl}	0.1	0.4	0.6	0.8	1.0
SAM	0.523	0.715	0.743	0.756	0.764
FAM	0.254	0.568	0.704	0.745	0.763

θ_{infl}	0%	1.5%	2.5%	5%	10%
SAM	0.999	0.840	0.743	0.505	0.208
FAM	0.919	0.785	0.704	0.413	0.137

σ_{infl}	0.01	0.02	0.03	0.04	0.05
SAM	0.780	0.763	0.743	0.721	0.696
FAM	0.780	0.764	0.704	0.602	0.497

Table 5: The effect of changing the three Vasicek model parameters (the mean-reversion rate κ_{infl}, the long-run mean rate of inflation θ_{infl}, and the volatility of inflation σ_{infl}) that govern the evolution of inflation, as shown in equations (3) and (4).

Finally, we consider the effect of changing the values of the Vasicek parameters that govern the evolution of the AA corporate bonds' yield curve. As with the Vasicek model parameters for inflation, both SAM and FAM increase when κ_{AA} increases or σ_{AA} decreases since, as before, this means less uncertainty. The effects here, however, are less extreme since the rate of inflation, especially over the long term, has a big effect on the amount owed, while many of the AA corporate bonds are bought at the beginning of the 30 year horizon before the effect of the random process is heavily felt. The higher the average instantaneous rate of return for the AA bonds, the better, so it is unsurprising to see both SAM and FAM increase as θ_{AA} increases.

κ_{AA}	0.1	0.3	0.5	0.7	0.9
SAM	0.713	0.740	0.743	0.744	0.744
FAM	0.666	0.701	0.704	0.704	0.704

θ_{AA}	1.5%	2.5%	3.5%	4.5%	5.5%
SAM	0.603	0.670	0.743	0.820	0.901
FAM	0.535	0.636	0.704	0.771	0.843

σ_{AA}	0.005	0.01	0.02	0.03	0.04
SAM	0.747	0.746	0.743	0.737	0.730
FAM	0.707	0.706	0.704	0.697	0.692

Table 6: The effect of changing the three Vasicek model parameters (the mean-reversion rate κ_{AA}, the long-run mean rate θ_{AA}, and the volatility σ_{AA})that govern the evolution of the instantaneous interest rate of AA corporate bonds, as shown in equation (5). We note that via equation (6) these parameters and the Vasicek model also yield the evolving value of AA corporate bonds with any coupon rate and maturity.

4.7 Monte Carlo for RFFR

To calculate RFFR, we need to model the cost of STRIPS, but this must be done in light of the fact that we have already modeled AA corporate bonds. To do this we define r_{Tr}, the instantaneous interest rate for treasury bonds, by

$$r_{Tr} = \frac{r_{AA}}{1+e^{-x}},$$

noting that this maintains the desired restriction $0 \le r_{Tr} \le r_{AA}$, regardless of the value of x. We model the value of x by a correlated Vasicek model process; specifically,

$$x(t+1) = x(t)e^{-\kappa_x} + \theta_x(1-e^{-\kappa_x}) + \sigma_x\sqrt{\frac{1-e^{-2\kappa_x}}{2\kappa_x}}\big(\rho_x Z_3 + \sqrt{1-\rho_x^2}Z_4\big),$$

where Z_3 is the same standard normal random variable used to compute r_{AA} in equation (5) and Z_4 is a standard normal random variable that is independent of Z_1, Z_2, and Z_3.

We are then able to use the Vasicek model parameters for treasuries to, as before, determine $ZCB_{Tr}(t,T)$, the cost of Treasury STRIPS (that is, zero-coupon bonds) at time t that will generate \$1 at time T, by using the formula

$$ZCB_{Tr}(t,T) = e^{B_{Tr}(t,T) - r_{Tr}(t)A_{Tr}(t,T)},$$

where

$$A_{Tr}(t,T) = \frac{1}{\kappa_{Tr}}\big(1-e^{-\kappa_{Tr}(T-t)}\big)$$

and

$$B_{Tr}(t,T) = \left(\theta_{Tr} - \frac{\sigma_{Tr}^2}{2\kappa_{Tr}^2}\right)\big(A_{Tr}(t,T) - T + t\big) - \frac{\sigma_{Tr}^2}{4\kappa_{Tr}}A_{Tr}^2(t,T).$$

The seven parameters in this model have been determined from short term AA corporate bond returns and short term treasury returns between 1996 and 2023, which give $\kappa_{\mathrm{x}} = 0.0158, \theta_{\mathrm{x}} = 1.011, \sigma_{\mathrm{x}} = 0.1328, \rho_{\mathrm{x}} = 0.57, \kappa_{\mathrm{Tr}} = 0.0698, \theta_{\mathrm{Tr}} = 0.0173$, and $\sigma_{\mathrm{Tr}} = 0.00673$. We have also chosen an initial x value of 1 and an initial instantaneous interest rate for treasuries of 2%.

With our model for STRIPS in place, we can compute RFFR at any time by altering the formula for the augmented funded ratio (or the funded ratio if preferred) to account for our using STRIPS. For example, the initial value for RFFR can be computed by altering our formula given in equation (2) for the initial augmented funded ratio to obtain:

$$\mathrm{RFFR} = \frac{80 + \sum_{t=1}^{5}(6-t) * ZCB_{\mathrm{Tr}}(0,t)}{\sum_{t=1}^{15} 5e^{i_{\mathrm{infl,estim}} * t} * ZCB_{\mathrm{Tr}}(0,t) + \sum_{t=16}^{30} 4e^{i_{\mathrm{infl,estim}} * t} * ZCB_{\mathrm{Tr}}(0,t)}.$$

This calculation, which uses different rates for each liability's time to reflect the interest rate changes for STRIPS over time, yields an initial RFFR value of 0.601. In contrast, the augmented funded ratio in equation (2) uses a fixed discount rate. In fact, Table 1 shows that a constant discount rate of 2.0% also corresponds to the augmented funded ratio of 0.601, so in this sense the changing STRIPS interest rate in this case "averages" to a constant discount rate of 2.0%.

As a rule of thumb, if RFFR reaches 1.3 at any time, an insurer will be willing take over the plan, meaning the owner has protected themselves from the risk of potential future defaults. To reflect this, in our RFFR Monte Carlo model, any time RFFR reaches 1.3, we assume the fund is sold to an insurer and therefore bankruptcy is no longer possible.

How important is this insurance option? From a bankruptcy point of view, it is actually of very little worth, because a plan whose RFFR is 1.3 is extremely unlikely to go bankrupt if we continue using the same fixed investment strategy that we used for SAM and FAM in the previous section. Consider the base case from before where $\alpha = 1$, but we can insure the plan if RFFR reaches 1.3. The chance of going bankrupt is shown by the orange curve in Figure 2. We note that at the end of the 30 years there is an 81.8% chance of going bankrupt with our insurance option, whereas the base case in Figure 1 without the insurance option has an 81.9% chance of going bankrupt after 30 years. In fact, comparing the orange curve in Figure 2 with Figure 1 shows less than a one percentage point difference at any time between the two figures. What if we allow the plan to be insured if RFFR reaches 1.1 instead of 1.3? Again, it makes no real difference. The chance of going bankrupt if the plan can be insured should RFFR reach 1.1 is shown by the blue curve in Figure 2, but it is nearly indistinguishable from the orange curve for the case where RFFR needs to reach 1.3. The differences only occur near the end of the 30-year horizon, and even then they are slight: the probability of going bankrupt is 81.4% instead of 81.8% at the end of the 30 years.

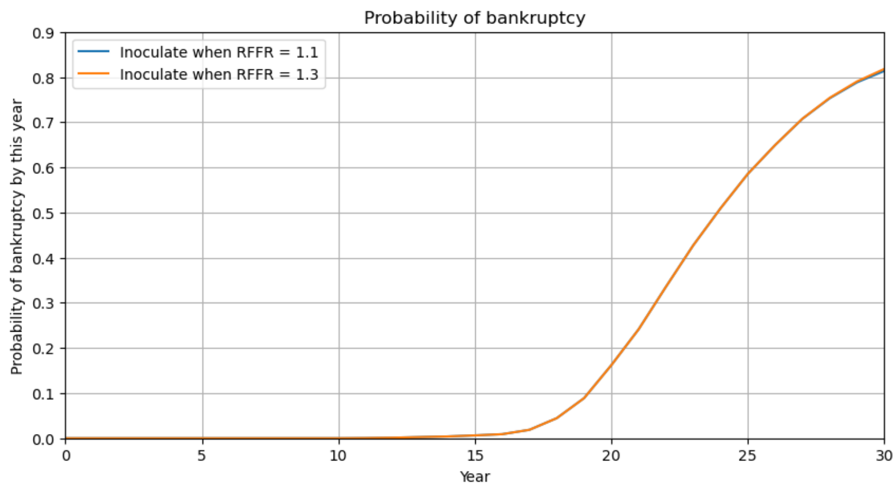

Figure 2: The cumulative probability over time of going bankrupt if the plan is inoculated from future bankruptcy any time RFFR reaches 1.3, versus any time RFFR reaches 1.1. The two curves are nearly indistinguishable, except at the end of the 30-year time period. Note that the asset and contribution multiplier, α, is 1 in these graphs.

There is another aspect to insuring a plan, however, and that is for financial planning from an accounting viewpoint. That is, once a plan is transferred to an insurer, the original owner need not worry about the plan any longer. Therefore, the earlier the plan is sold, the better. From this perspective, the insurance option is desirable, and we note from Figure 3 that being able to inoculate the plan when RFFR reaches 1.3 takes longer than if it can be inoculated when RFFR reaches 1.1. For example, there is an 8% chance of being inoculated after 20 years if we require RFFR to reach 1.3, but this increases to 12% if we only require RFFR to reach 1.1.

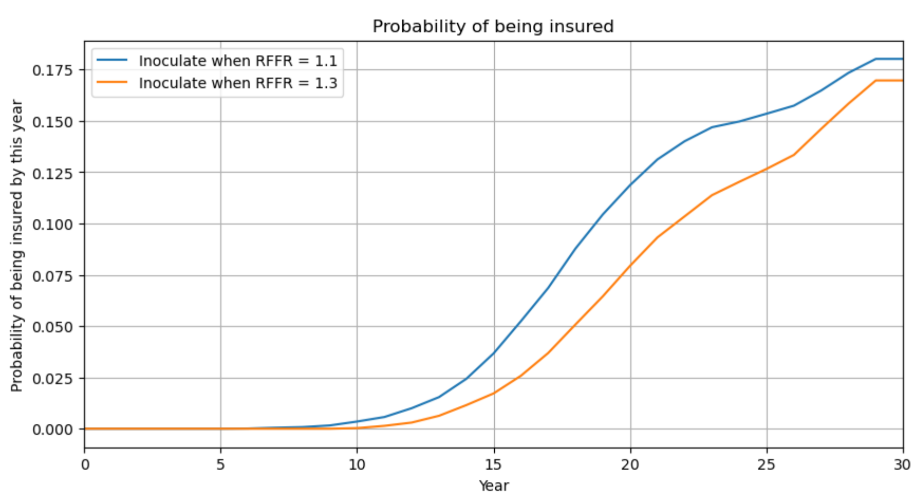

Figure 3: The cumulative probability over time of being able to inoculate the plan from future bankruptcy any time RFFR reaches 1.3, versus any time RFFR reaches 1.1. Again, note that $\alpha = 1$ in these graphs.

The timing of obtaining insurance is everything here. As we reach the end of our liabilities horizon, the amount of future liabilities diminishes significantly, causing the denominator in the formula for RFFR to decrease, and therefore RFFR increases. It is therefore rare that a plan neither goes bankrupt nor reaches an RFFR of 1.3 eventually. In fact, there's only a 1.23% chance of this happening, which goes down to a 0.60% chance if RFFR only needs to reach 1.1 instead of 1.3. That is, having a plan be insurable just a few years before its end is not particularly useful, even from an accounting point of view.

We therefore must present the full graph of the cumulative probability of being inoculated over time as the result of any RFFR analysis. In Figure 4, we see these full graphs as we increase the value of α from 1, where bankruptcy is likely, to higher values of α, where the likelihood is diminished and inoculation becomes more likely. As before, the likelihood of remaining neither bankrupt nor eventually able to reach an RFFR of 1.3 remains small, never exceeding 2.5% in any of the five cases presented.

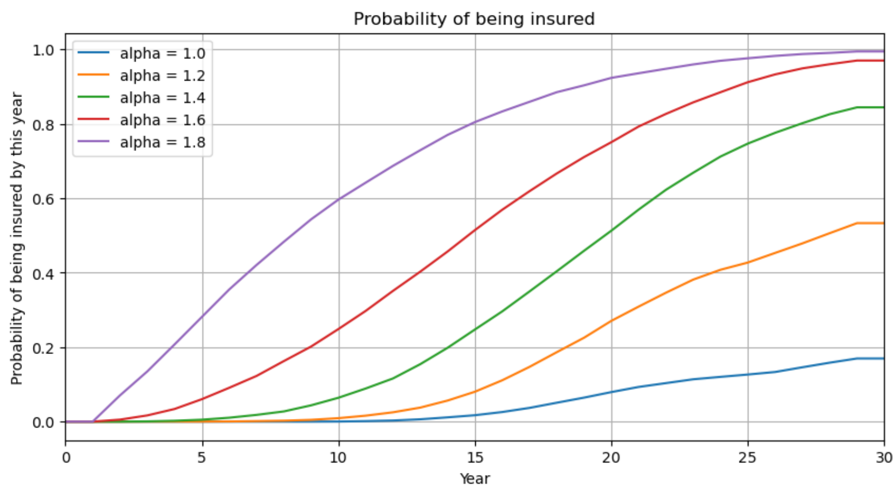

Figure 4: The cumulative probability over time of being able to inoculate the plan from future bankruptcy, given various values of the asset and contribution multiplier (α), assuming inoculation occurs any time RFFR reaches 1.3.

Because the liabilities are affected by inflation, RFFR must use $i_{\text{infl,estim}}$ to estimate the future liabilities before using STRIPS to find their present value. Therefore, $i_{\text{infl,estim}}$ can have a significant effect on the value of RFFR and whether of not early inoculation is possible. In Figure 5, we can see the importance of $i_{\text{infl,estim}}$. In particular, we have set $\alpha = 1.8$ and shown the effect of changing $i_{\text{infl,estim}}$ from its base case value of 3% to 2% or to 4%. Comparing these three results shows, for example, that there is a 60% chance of inoculating the plan by year 3 if $i_{\text{infl,estim}} = 2\%$, but this is delayed to year 10 if $i_{\text{infl,estim}} = 3\%$, and then to year 14 if $i_{\text{infl,estim}} = 4\%$.

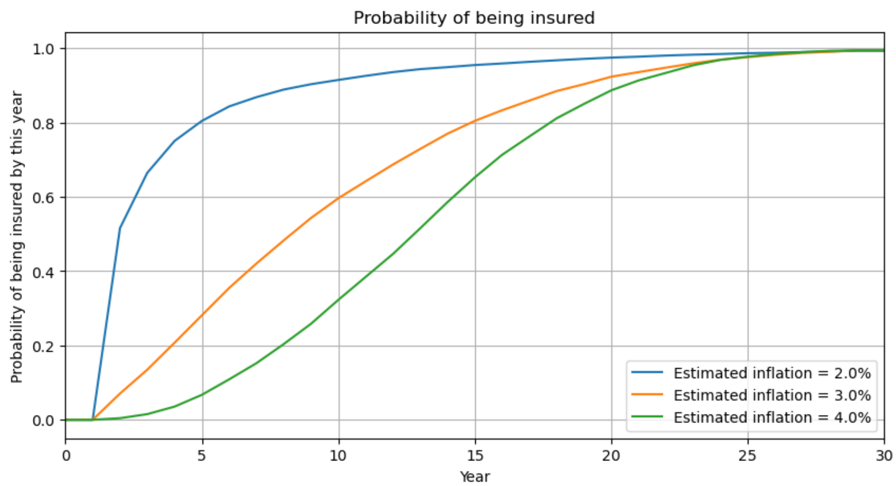

Figure 5: Seting $\alpha = 1.80$ and then changing the estimated inflation, $i_{\text{infl,estim}}$, from its value in the base case (used in Figure 4), which is 3%, to 2% and then to 4%. The middle curve in this figure (for the base case, $i_{\text{infl,estim}} = 3\%$) is identical to the top curve in Figure 4 for $\alpha = 1.80$.

Because the timing of the plan inoculation is the key result of an RFFR analysis, it is far from clear what it would mean to determine an optimal strategy in an RFFR context. That is, in our example, we have the probability of being insured at each of 29 times. Any optimization would have to weight all 29 probabilities to create one quantity to optimize, and it is far from clear what weighting would be meaningful in this context. Further, if we only want to optimize the eventual probability of being inoculated, we have already seen that this is essentially equivalent to maximizing the probability of remaining solvent, and that is equivalent to maximizing SAM. Since both SAM and FAM are single numbers that can be maximized, the notion of finding investments to maximize SAM or FAM makes intuitive sense, and we show how to accomplish this in the next section.

5 Determining the Optimal Investment Strategy to Maximize SAM or FAM using Dynamic Programming

The Monte Carlo analysis of the previous section works quite well for any fixed investment portfolio strategy, but Monte Carlo methods are completely impractical for determining an optimal dynamic investment portfolio strategy. To determine this strategy, which chooses an optimal investment portfolio in light of the fact that a new investment portfolio can be chosen in each future year, we use dynamic programming. We will see that dynamic programming can compute the optimal dynamic investment portfolio strategy for SAM or FAM within a few minutes.

As before, SAM is the reciprocal of the minimum α value needed to keep the chance of bankruptcy below a specified probability. However, in our context of portfolio optimization in this section, it is the reciprocal of the minimum α value *given* that the liability based plan can alter its investment portfolio each year. That is, for each α value considered, we determine the optimal dynamic investment portfolio strategy that minimizes the probability of failing to pay even one of the plan's liabilities. In particular, this optimal strategy specifies the investment portfolio that the liability based plan must optimally select, which is a function of time and of the portfolio's remaining wealth at that time. This optimal investment portfolio strategy can be computed for any α in a few seconds via single goal dynamic programming, as shown in Das et al. (2020).

Dynamic programming can also be used to determine the optimal investment portfolio strategy for FAM. However, this is done via the Efficient Goal Probability Frontier method (shown in Das et al. (2023)), which uses dynamic programming for multiple goals (shown in Das et al. (2022)), instead of a single goal. The Efficient Goal Probability Frontier method requires a longer computational time. For instance, in our examples below, SAM and FAM each required computations at about 8 different values of α. Running on a home computer with an Apple M1 Max chip, the SAM calculation took a total time of just over half a minute, while the FAM calculation took two minutes and 15 seconds.

For our examples, we will use much of the basic set-up that was used in the previous section. As in that section, we start with 80MM. Over the first five years, there will be contributions of \$5MM, \$4MM, \$3MM, \$2MM, and \$1MM nominal dollars, but nothing after that. The plan's liabilities require it to pay \$5MM present-valued dollars (adjusted for inflation) in each of the first 15 years, followed by \$4MM present-valued dollars in each of the 15 years after that.

Because dynamic programming determines the optimal strategy by working backwards in time, it must rely on state variables to describe the forward evolution of a portfolio up to any given time. However, we must limit the number of state variables in our model to avoid the so-called "curse of dimensionality," which would quickly slow the computation down considerably. We therefore use just the time and the current portfolio's worth as our state variables. Instead of adding an additional state variable for inflation, we simply assume an inflation rate of 3%. It is impossible to keep track of all previous bond purchases as we did in the previous section using Monte Carlo simulation, so we will instead use a simple bond fund. Further, we will assume the bond component's price evolves by geometric Brownian motion instead of evolving by the Vasicek model since the Vasicek model would require an additional state variable for the bond fund's instantaneous interest rate. Should our assumption regarding the interest rate being constant or our assumptions regarding the bond model be considered problematic, we suggest first applying the method presented in this section with these approximations so as to determine an optimal strategy, followed by applying the method presented in the previous section, using this (now fixed) optimal strategy.

For many LDI funds or other liability based plans that are restricted to being invested very conservatively, there may be little variation allowed in the plan's investments, in which case portfolio investment optimization may of little to no use. In these cases, the Monte Carlo approach of the previous section may be sufficient. However in other liability based plans, including some LAI plans and many ALM plans, there is a wider range of permissible investments. Because we assume a geometric Brownian motion model for both the stocks and the bonds here, any optimal investment portfolio must lie on the efficient frontier. In Table 7, we give an example with 21 expected return (μ) and volatility (σ) pairs that lie on a predicted efficient frontier governing stock and bond portfolios. Our dynamic programming algorithms show how to optimally move among these 21 investment portfolios, which are numbered from 0 (very conservative, all bonds) to 21 (very aggressive, all stock), changing in 5% increments, so portfolio 1 is 95% bonds and 5% stock, portfolio 2 is 90% bonds and 10% stock, etc.

Portfolio number	0	1	2	3	4	5	6	7	8	9
Percent bonds	100%	95%	90%	85%	80%	75%	70%	65%	60%	55%
Percent stocks	0%	5%	10%	15%	20%	25%	30%	35%	40%	45%
Expected return (μ)	0.046	0.048	0.05	0.052	0.054	0.056	0.058	0.06	0.062	0.064
Volatility (σ)	0.0439	0.0444	0.0462	0.0491	0.053	0.057	0.063	0.068	0.074	0.081

Portfolio number	10	11	12	13	14	15	16	17	18	19	20
Percent bonds	50%	45%	40%	35%	30%	25%	20%	15%	10%	5%	0%
Percent stocks	50%	55%	60%	65%	70%	75%	80%	85%	90%	95%	100%
Expected return (μ)	0.066	0.068	0.07	0.072	0.074	0.076	0.078	0.08	0.082	0.084	0.086
Volatility (σ)	0.087	0.094	0.101	0.107	0.114	0.121	0.128	0.136	0.143	0.150	0.157

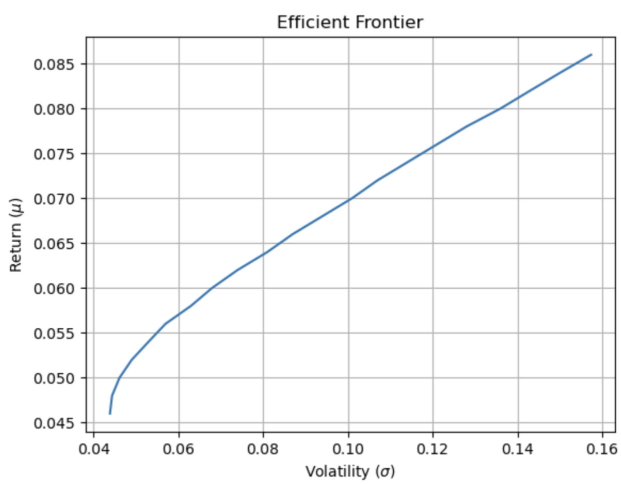

Table 7: The top table describes the 21 investment portfolios that are available to the investor, where the portfolio's expected return and volatility increases with the portfolio number. These investment portfolios lie along the efficient frontier shown in the bottom panel.

We first calculate SAM for our example under the requirement that the chance of bankruptcy does not exceed 10%. The optimized investment portfolio strategy determined by our dynamic programming approach is given in the left panel of Figure 6, which corresponds to a SAM value of 0.868. In contrast, for FAM, we require the probability of not paying each of the liabilities in the first ten years to be no higher than 0.5%, then, for each of the next ten years, no higher than 5%, and finally, for each of the last 10 years, no higher than 10%. The optimized investment portfolio strategy determined by our multiple goals dynamic programming approach is given in the right panel of Figure 6, which corresponds to a FAM value of 0.973.

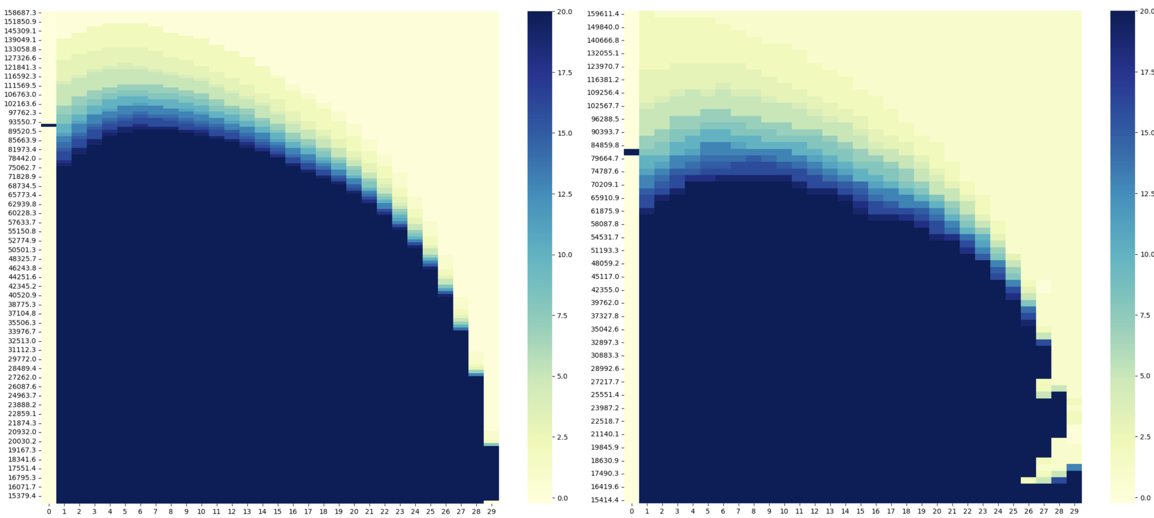

Figure 6: The optimal investment portfolio strategy for our SAM example (left panel) and our FAM example (right panel). The optimal investment portfolio is a function of time, in years, (on the horizontal axis) and the portfolio's worth, in dollars, (on the vertical axis). The darker the color, the more aggressive the optimal investment portfolio.

The restrictions in SAM are generally harder to satisfy than the restrictions in FAM, because SAM requires every liability is met while FAM only requires that the probability of satisfying each liability be above specified (usually high) levels. Therefore, it is not surprising that the value for SAM in our case is smaller than the value for FAM. This also explains why we can see from Figure 6 that the optimized dynamic investment portfolio strategy is more aggressive for SAM than for FAM. Because the uncertainty for how conservative or aggressive we will later need to be decreases over time, in both of the panels in Figure 6 we see the spread in the optimal investment portfolios diminish as time progresses. In the FAM panel on the right, the conservative portfolio "notches" in later years correspond to cases where being more conservative safeguards the ability to pay liabilities for one more year (the large, bottom notch) or two more years (the small, top notch). More specifically, within these notches, more aggressive portfolios' ability to potentially pay for additional years of liabilities do not outweigh the potential risk of losing the ability to pay the liabilities in years that a conservative portfolio can guarantee.

We note that our values for SAM and FAM are higher in this section than in the Monte Carlo case presented in Section 4. There are a number of reasons for this.

One reason is obvious: we are using dynamic investment portfolio optimization in this section, and we had a fixed investment portfolio in Section 4. Another reason is that in our Monte Carlo case, the expected returns of the bonds and of the stocks were lower than in this section and the volatilities were higher. Further, inflation was stochastic in the previous section, even though the average inflation rate in that section was lower than the assumed inflation rate in this section. Finally, the different notion for FAM used in Section 4 involved more difficult restrictions to satisfy than the normal notion for FAM used in this section. We note that all of these effects were strong enough to overcome the fact that SAM had a 20% bankruptcy limit in the previous section, and only a 10% bankruptcy limit here.

How much does investment portfolio optimization by itself improve SAM and FAM? If some of the investment portfolio options are considered too aggressive to be considered, we can simply remove them and rerun our dynamic programming algorithms. In Table 8, we show the effect on SAM and FAM of keeping progressively fewer aggressive investment portfolios available. The effect of progressively removing aggressive investment portfolio options on the optimal investment portfolio strategies that were shown in Figure 6 is straightforward: the parts of the figure that correspond to aggressive investment portfolio options that have been removed simply now take the most aggressive investment portfolio option still available.

Portfolios numbers	0–20	0–18	0–16	0–14	0–12	0–10	0–8	0–6	0–4	0–2	0	20	0 or 20
SAM	0.868	0.862	0.859	0.848	0.834	0.823	0.821	0.798	0.785	0.760	0.731	0.692	0.823
FAM	0.973	0.964	0.950	0.924	0.919	0.912	0.901	0.895	0.862	0849	0.792	0.779	0.898

Table 8: All but the last two columns show the effect on SAM and FAM when we progressively restrict the potential investment portfolios to more and more conservative (lower numbered) options. The second to last column corresponds to being restricted to all stock (portfolio 20). The last column corresponds to being restricted to a portfolio strategy of jumping between either all bonds (portfolio 0) or all stock (portfolio 20).

In the third to last column of Table 8, we look at the effect of having all bonds, so there is no investment portfolio optimization. We note that it is considerably lower than the case where we have optimized the use of all 21 investment portfolios. In the second to last column, we consider the opposite case where we have all stock, which leads to even worse results. This is due to having to overcome the significant negative effect of the stock position's volatility on being able to maintain high probabilities of meeting liabilities. Finally, while Figure 6 shows that at many times and wealth values it is optimal to have either all bonds or all stock, the figure does not take into account the likelihood of being at the time and wealth values that correspond to intermediate investment portfolios. We therefore computed the final column, which optimizes the investment portfolio strategy if we can only be all in bonds or all in stock. We note that there is still a significant reduction in SAM and FAM versus the case with access to all 21 investment portfolios. That is, Table 8 gives a strong sense for the considerable amount of improvement to SAM and FAM that dynamic investment portfolio optimization can provide.

6 Conclusions

The funded ratio of a liability based plan (LBP) has the advantage of having a simple definition, but that simplicity comes with considerable weaknesses if it is not augmented by additional measures for the LBP's health. In particular, the funded ratio lacks an ability to directly take a number of important factors into account, including future contributions, employers' financial health, the yield curve not being constant, and the effect of the timing for when contributions occur versus when liabilities are due. Chief among the funded ratio's weaknesses, however, is its inability to take into account stochastic factors that are key to understanding the true health of an LBP.

In this paper we introduce three additional measures that help reach a more complete understanding of the health of an LBP. One measure, the risk-free funded ratio (RFFR), is a market based measure that is particularly helpful in determining if an LBP should be bought or sold. The other two measures, the solvency asset multiple (SAM) and the funded asset multiple (FAM), are heavily probability based measures that correspond to the reciprocal of the multiple of the current funds and future contributions needed to attain specified probabilities of remaining solvent at the end of the LBP's horizon (SAM) or attaining specified probabilities of satisfying each years' liabilities (FAM).

SAM and FAM are able to take into account many factors that the funded ratio cannot address by its nature, including stochastic factors. Because SAM and FAM are probability based, while the funded ratio is not, SAM and FAM often can provide crucial information that is key to guiding a company's internal policy discussions regarding their LBPs. For example, consider the common, but also questioned, rule of thumb that LBPs with funded ratios above 0.8 are generally healthy. In this paper, we presented a simple case where the funded ratio would generally be considered to be within this "healthy" range, but from the probabilistic perspective, we showed there is an 81.9% chance that this LBP will go bankrupt without additional funding. Using SAM or FAM gives a true sense of how much additional funding is necessary to attain a specific probability of maintaining solvency in all years (SAM) or to attain sufficiently high probabilities of meeting liabilities in each year (FAM).

The funded ratio gives no sense of how investments should be optimally selected to maximize the probability of meeting an LBP's liabilities. In Chapter 5, however, we showed how SAM and FAM can be used to accomplish this. This ability is particularly important if we are allowing for a selection of investments over a wide array of expected returns and volatilities.

Because our method in Chapter 5 uses dynamic programming, the "curse of dimensionality" restricts the number of stochastic factors it can accommodate. Methods like reinforcement learning (RL), however, offer hope in the future for enabling optimization with models containing multiple stochastic factors. Examples of using RL to help optimize financial models include Hambly et al. (2021); Duarte et al. (2021); and Chen et al. (2021). In addition to RL approaches, there are other simulation-based approaches like Blay et al. (2020). However, the financial context of each of these four cited papers differs from the LBP goal questions examined in this paper.

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