Pricing Interest Rate Derivatives: A General Approach

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The relationship between affine stochastic processes and bond pricing equations in exponential term structure models has been well established. We connect this result to the pricing of interest rate derivatives. If the term structure model is exponential affine, then there is a linkage between the bond pricing solution and the prices of many widely traded interest rate derivative securities. Our results apply to m-factor processes with n diffusions and l jump processes. The pricing solutions require at most a single numerical integral, making the model easy to implement. We discuss many options that yield solutions using the methods of the article.

The literature on term structure modeling has evolved from one-factor diffusion models such as Vasicek (1977) and Cox, Ingersoll, and Ross (1985) to multifactor models such as Brennan and Schwartz (1977), Duffie and Singleton (1997), Longstaff and Schwartz (1992), and Balduzzi, Das, and Foresi (1998), as well as jump-diffusion models such as Ahn and Thompson (1988), Bakshi and Madan (2000), Das and Foresi (1996), Das (1998), and Duffie, Pan, and Singleton (2000). The motivation for this evolution in term structure models has come from empirical articles such as Chan et al., Sanders (1992) and Ait-Sahalia (1996).[1] However, as work proceeds on better matching the dynamics of the short rate to the observed term structure, the area of fixed income derivative pricing, the main application for modeling the short rate process, has lagged behind. In this article we attempt to bridge the gap between the multifactor, jump-diffusion models of the short rate that are commonly used and the pricing of fixed income derivatives. Specifically

This is a substantially revised version of the article "Pricing Average Interest Rate Options: A General Approach" (1998) based on the original article "Average Interest" (NBER Working Paper no. 6045 [1997]). A substantial part of the work of the second author was done while at Harvard University and the University of California, Berkeley. We are indebted for the many helpful comments of the editor and an anonymous referee who have helped tremendously in improving the content and exposition of the article. Thanks also to Marco Avellaneda, Steven Evans, Eric Reiner, Rangarajan Sundaram, Vladimir Finklestein, Alex Levin, Jun Liu, Daniel Stroock, and seminar participants at the Courant Institute of Mathematical Sciences, New York University, the Computational Finance Group at Purdue University, University of Texas, Austin, and the Risk99 conference for their comments. Address correspondence to the authors at Harvard University, Graduate School of Business Administration, Morgan Hall, Soldiers Field, Boston, MA 02163.

[1] Many other articles including those by Brown and Dybvig (1989), Litterman and Scheinkman (1991), and Stambaugh (1988) have similar conclusions.

The Review of Financial Studies Spring 2002 Vol. 15, No. 1, pp. 195–241

we show that *any* interest rate process (with any number of factors including stochastic volatility, stochastic central tendency, etc., or utilizing diffusion or jump-diffusion processes) that leads to an exponential term structure model also lends itself to analytic solutions for *three* large classes of fixed income securities. These methods support numerical techniques which allow for easy implementation in the context of a no-arbitrage approach. It is our hope that the results of this article will allow both researchers and practitioners to focus on the appropriate stochastic process for the short rate and its factors, and obviate concerns as to whether specific forms of the short rate lead to tractable solutions for popular fixed income securities.

The benchmark article of Duffie and Kan (1996) established the link between affine stochastic processes and exponential affine term structure models.[2] They showed that the factor coefficients of these term structure models are solutions to a system of simultaneous Riccati equations and that these coefficients are functions of the time to maturity. The kernel of our technique resides in the fact that the solution for different types of interest rate options solves an almost identical system of equations. The only difference between the two sets of equations is in the constant terms underlying the equations. By manipulating the Riccati equations and varying the constant terms, we develop a procedure to price options using the known solution components of the original term structure model. Thus we essentially show that once the exponential affine term structure model is derived, the pricing formulas for a wide range of popular fixed-income derivatives can be written by inspection from the components of the term structure model.[3] Specifically, we show that this approach is feasible for three large classes of fixed income derivatives: those with (1) payoffs that are linear in the short rate and factors; (2) payoffs that are exponential affine in the short rate and factors; and (3) payoffs that are an integral over time of a linear combination of the short rate and factors. These three payoff structures encompass many popular fixed-income derivatives.

Our technique is general in that it applies to any multifactor, *exponential affine* term structure model with multiple Wiener and jump processes. No matter how many jump-diffusion stochastic processes are used, for standard derivatives, our approach involves evaluation of at most two one-dimensional integrals, resulting in easy computation. Furthermore, in the final section of this article we show that the techniques can be easily extended beyond the exponential affine class to the class of term structure models we call "exponential separable" models, such as those of Constantinides (1992) and Longstaff (1989). In addition, we also show how to utilize the results of the

[2] Dai and Singleton (1997) provide a characterization of the exponential affine class of term structure models as they unify and generalize this class.

[3] See also Duffie, Pan, and Singleton (2000) and Bakshi and Madan (2000), who developed results that parallel some of those derived in this article.

article in the context of no-arbitrage models, such as those of Hull and White (1990) and Black, Derman, and Toy (1991), which allow for exact calibration with observed data.

To demonstrate the technique we provide closed-form solutions for options under a jump-diffusion model. We price options on bonds, futures, and interest rate caps and floors, since these are the most common forms of term structure derivatives. We also price options on average interest rates, in order to demonstrate a parsimonious approach based on expansion of the state space.[4] An important tool in our approach is the use of Fourier inversion methods as in Heston (1993).[5]. Though the results of this article pertain to term structure models, the techniques provided extend to several other market settings.

The plan of the article is as follows. In Section 1 we specify the interest rate process and the term structure model. We then introduce the pricing technology for fixed income securities that have general payoff functions of the interest rate process. We proceed in Sections 2–4 to develop analytic solutions, in terms of the components of the term structure model in Section 1, for the three categories of derivative payoff functions considered in the article: linear payoffs in the state variables are handled in Section 2, exponential affine payoffs in Section 3, and integro-linear (or a payoff function that is an integral over time of a linear combination of the factors) payoffs are dealt with in Section 4. The details of these derivations are described in the appendix, which contains many analytical features of interest. Section 5 discusses model implementation. Section 6 provides examples of the procedures laid out in the article. Section 7 presents extensions and Section 8 concludes.

1. Generalized Option Valuation

In this section we present the setup for the general valuation principles in the article. We specify the general interest rate process and the term structure model for which we will be able to derive general option valuation formulas. The restrictions on the interest rate dynamics imposed here are the same as those specified in Duffie and Kan (1996) for jump-diffusion processes. These restrictions lead to an exponential affine term structure model. With the aid of the Feynman–Kac theorem, which is stated below, we derive a

[4] The idea very simply is to expand the state space from that of a traditional Black and Scholes (1973) and Merton (1973) setup with m-state variables to $m+1$-state variables where the additional variable is the average (i.e., arithmetic integral) of the underlying. Bakshi and Madan (2000) provide a spanning analysis of this idea.

[5] Fourier methods have been used subsequently in many articles in finance including Eydeland and Geman (1994), Scott (1995), Bates (1996), Chacko (1996, 1998), Das and Foresi (1996), Bakshi, Cao, and Chen (1997), Bakshi and Madan (1997, 2000), Singleton (1997), and Duffie, Pan, and Singleton (2000), Davydov and Linetsky (1999), Heston and Nandi (1999), Jagannathan and Sun (1999), Leblanc and Scaillet (1998), Levin (1998), Van Steenkiste and Foresi (1999). Bakshi and Madan (1997, 2000) link Fourier transform methods to a state-price framework, while Duffie, Pan, and Singleton (1998) describe the application of these techniques to problems in the area of equity, interest rate and default risk options. Van Steenkiste and Foresi (1999) show how to derive state prices in the same general framework and apply Fast–Fourier methods to price American options.

general valuation equation for fixed-income securities. Our task in subsequent sections will be to solve this equation for large classes of fixed-income securities using only the components of the term structure model presented in this section.

1.1 Interest rate dynamics

The economy is a continuous-trading economy with a trading interval $[0, T]$ for a fixed $T > 0$. The uncertainty in the economy is characterized by a probability space $(\Omega, \mathcal{F}, Q)$ which represents the risk-neutral probability measure in the economy.[6] Dynamic evolution of the system over time takes place with respect to a filtration $\{\mathcal{F}(t) : t \geq 0\}$ satisfying the conditions in Protter (1990, chap. V).

Let $\mathbf{N}_t$ represent a vector l of orthogonal Poisson processes, and let $\mathbf{W}_t$ represent a vector of n Wiener processes. Each Poisson, or jump process, can be thought of as a counter. When a jump occurs, the jump process increments upward by 1 unit. The jump intensities of the Poisson processes are given by $\lambda_i \geq 0$, $i = 1 \ldots l$, and are constant over $[0, T]$.

The term structure of zero-coupon bond prices is formed from the instantaneous interest rate and a set of m factors in the economy. The risk-neutral processes governing the interest rate and the factors are given by a vector of strong Markov processes:

$$dr_t = \mu(r_v, \mathbf{x}_v)\, dt + \sigma'(r_v, \mathbf{x}_v)\, d\mathbf{W} + \mathbf{J}_r'\, d\mathbf{N} \tag{1}$$

$$d\mathbf{x}_t = \alpha(\mathbf{x}_t)\, dt + \delta(\mathbf{x}_t)\, d\mathbf{W} + \mathbf{J}_x\, d\mathbf{N} \tag{2}$$

The $m \times 1$ vector, $\mathbf{x}_t$, represents a set of Markov factors which influence the marginal productivity of capital, and thus the interest rate, in the economy. We assume that the parameters of the drift, diffusion, and jump coefficients in the SDE are bounded [in the sense of Gihman and Skorohod (1979, pp. 128–130)], and are such that a unique, strong solution to Equation (1) exists [the conditions in Pardoux (1997) are met] [see also Kurtz and Protter (1996a, b)]. The magnitudes of the Poisson processes are defined by the $l \times 1$ vector $\mathbf{J}_r$ and the $m \times l$ matrix $\mathbf{J}_x$ of correlated random variables. It is assumed that the conditional distribution of the jump size is independent of the state variables.

We assume that the instantaneous diffusion covariance matrix of the state variables is given by $\boldsymbol{\Lambda}(\mathbf{x}_t)$, while the vector of instantaneous diffusion covariances between the state variables $x_{i,t}$, $i = 1 \ldots m$, and r_t is given by $\rho(\mathbf{x}_t)$.

1.2 Fundamental principles

Our derivations of solutions for the models explored in this article use the Feynman–Kac theorem[7] and the Fourier inversion theorem extensively. In

[6] Unless indicated otherwise, all computations reported in the article are with respect to the risk-neutral probability measure and not the objective probability measure.

[7] See Duffie (1996) for more details regarding the Feynman–Kac relation.

this section we state these for reference and establish the conditions for their applicability.

Definition 1. (*Feynman–Kac*). *For any variable X (satisfying the regularity conditions stated below) determined by a stochastic differential equation of the form*

$$dX_t = \alpha_x(X_t)\,dt + \delta_x(X_t)\,dW + J(X_t)\,dN(\lambda),$$

the solution, $F(X_t)$, to the expression

$$E_t\left[e^{-\int_t^T g(X_v)\,dv} f(X_T)\right],$$

where $f, g \in \mathscr{C}^{2,1}$, is determined by the equation

$$\mathscr{D}F(X_t) = g(X_t)F,$$

where $\mathscr{D}$ is the differential operator defined by

$$\mathscr{D}F(X_t) = \frac{1}{2}\delta_x^2 \frac{\partial^2 F}{\partial X_t^2} + \alpha_x \frac{\partial F}{\partial X_t} - \frac{\partial F}{\partial \tau} + \lambda E_{t-}[F(X_t + J) - F(X_t)].$$

The boundary condition for this partial differential difference equation is given by $F(X_T) = f(X_T)$.

This is simply the univariate version of the differential operator (see note 8 for the multivariate version). We will utilize multivariate versions of this theorem throughout the article. To apply Feynman–Kac, certain regularity conditions must be satisfied by the underlying processes. We require in this article that the processes given in Equations (1) and (2) satisfy these conditions, which is stated in univariate form in the following definition.

Definition 2. (*Conditions for Feynman–Kac*). *The process X_t in Remark* (1) *must satisfy the necessary growth and Lipshitz conditions.* [*See Duffie* (1996, *pp.* 292–295) *or Karatzas and Shreve* (1988, *p.* 366) *for explicit details. Additional details are available from the exposition in Duffie, Pan, and Singleton* (1998).] *The conditions required are*

1. *The functions in question, that is, f, g, α, δ, F are continuous.*
2. *The polynomial growth condition is satisfied*: $|F(X_t, t)| \leq A(1 + \|X\|^q)$, *for some constants* $A, q > 0$.
3. $f(X_t) \geq 0$, *or it satisfies a polynomial growth condition in X_t.* [*Conditions* 2 *and* 3 *are either/or type, see Karatzas and Shreve* (1988, *condition* (7.10), *p.* 366).]
4. *For the jump, we will require that the jump transform* $\int_{\mathbb{R}} \exp(cv)\,dJ(v)$, $c \in \mathscr{C}$ *be well defined.*

We also utilize a version of the Fourier inversion formula that relates the cumulative distribution of a density function to its Fourier transform. In the article, we regularly solve for the Fourier transform of a density and invert this transform to get at the cumulative distribution function, which is needed for security pricing.

Theorem 3. (*Fourier Inversion*). *If the density function f satisfies the following conditions* [*see Shephard* (1991)],

1. *f is integrable in the Lebesgue sense, that is, $f \in L$.*
2. *Its characteristic function is well defined as $\varphi(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f(x)\,dx$, and is integrable* [*and the characteristic is "well behaved" in the sense defined by Duffie, Pan, and Singleton* (1998)].
3. *$\Delta[\varphi(\omega)\exp(-i\omega x)/i\omega]$ is uniformly bounded, where $\Delta\eta(\omega) \equiv \eta(\omega) + \eta(-\omega)$, then the probability function can be obtained by Fourier inversion,*

$$F(x) = \frac{1}{2} - \frac{1}{2\pi}\int_0^{\infty} \Delta_\omega \left[\frac{\varphi(\omega)e^{-i\omega x}}{i\omega}\right].$$

1.3 Bond prices

With the risk-neutral interest rate process known, we can write an expression for the price of any traded security in the economy. Specifically, let $P_t(r, \mathbf{x}; \tau)$ represent the price at time t of a security that matures after a period of time τ. Then, we have the following partial differential difference equation (PDDE) for the price of a bond [Black and Scholes (1973), Merton (1973), Courtadon (1982), Cox, Ingersoll, and Ross (1985), see Ahn and Gao (1999)]:

$$0 = \mathcal{D}P_t + \mathbf{d}\ \mathbf{s}'P_t \tag{3}$$

where $\mathbf{d}$ is a row vector of constants and $\mathcal{D}$ represents the usual differential operator.[8] $\mathbf{s} = [r_t, \mathbf{x}_t, 1]$ is a row vector comprising the current levels of the short rate and the factors, and an additional parameter required for special forms of payoff functions.

In the case of traded securities, $\mathbf{d} = \mathbf{d}^* \equiv [-1, \mathbf{0}, 0]$, but we will assume for now that $\mathbf{d}$ is an arbitrary vector of constants. We do this because in subsequent sections we will utilize transformations to Equation (3) where partial differential equations with $\mathbf{d} \neq \mathbf{d}^*$ will result. For a zero-coupon bond that pays off \$1 at maturity, the boundary condition that is satisfied by Equation (3) is $P(\tau = 0) = 1$.

[8] The differential operator applied to a function P_t is defined as

$$\begin{aligned}\mathcal{D}P_t = {} & \frac{1}{2}\sum_{i=1}^{n}\sigma_i^2(r_t, \mathbf{x}_t)\frac{\partial^2 P_t}{\partial r^2} + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\Lambda_{ij}(\mathbf{x}_t)\frac{\partial^2 P_t}{\partial x_i \partial x_j} + \sum_{i=1}^{m}\rho_i(\mathbf{x}_t)\frac{\partial^2 P_t}{\partial r \partial x_i} \\ & + \mu(r, \mathbf{x}_t)\frac{\partial P_t}{\partial r} + \sum_{i=1}^{m}\alpha_i(\mathbf{x}_t)\frac{\partial P_t}{\partial x_i} + \frac{\partial P_t}{\partial t} \\ & + \sum_{i=1}^{l}\lambda_i E[P(r + J_{r,i}, \mathbf{x}_t + \mathbf{J}_x^{(i)}; \tau) - P(r, \mathbf{x}; \tau)],\end{aligned}$$

where the expression $\mathbf{J}_x^{(i)}$ represents a modified version of the matrix $\mathbf{J}_x$ of jump magnitudes. $\mathbf{J}_x$ is modified so that all but the ith column of $\mathbf{J}_x$ is zero. This is a direct consequence of Ito's lemma.

As is well known, we can use the Feynman–Kac theorem to write the solution to Equation (3) as

$$P(\tau) = E_t\left[e^{-\int_t^T r_v\,dv}1\right],$$

which is the standard discounted form for a discount bond price.

We need to impose restrictions on the drift and diffusion terms in Equations (1) and (2) in order to ensure that the solution to Equation (3) is exponential affine. From Duffie and Kan (1996), we know that the term structures of zero-coupon bond prices are of the exponential affine class, that is, those of the form

$$P_t(\tau) = \exp\left[A(\tau)r_t + \sum_{i=1}^{m} B_i(\tau)x_i + C(\tau)\right], \tag{4}$$

where $A(\tau)$, $B_i(\tau)$, $i = 1, \ldots, m$, and $C(\tau)$ are functions of time to maturity only if the drift terms and the square of each diffusion term of Equations (1) and (2) are linear in the interest rate and the factors, and if the jump magnitudes of Equations (1) and (2) have linear (in the interest rate and the factors) moment-generating functions.

The solutions to these functions are each determined by a separate system of ordinary differential equations (ODEs). Associated with each ODE is a unique boundary condition. For the remainder of the article we will need to be concerned only with the cases where the boundary conditions for Equation (3) are such that resulting boundary conditions on $A(\tau)$, $B_i(\tau)$, $i = 1, \ldots, m$, and $C(\tau)$ are given by

$$\begin{aligned} A(0) &= a \\ B_1(0) &= b_1 \\ &\vdots \\ B_m(0) &= b_m \\ C(0) &= c, \end{aligned} \tag{5}$$

where $a, b_1, \ldots, b_m, c$ are constants. In the case of zero-coupon bond prices, the exponential affine form of these prices allows us to break up Equation (3) using the well-used technique of separation of variables into a set of Riccati equations for $A(\tau)$, $B_i(\tau)$, $i = 1, \ldots, m$, and $C(\tau)$. If the boundary conditions for the system of Riccati equations are given by Equation (5), then the solutions to $A(\tau)$, $B_1(\tau), \ldots, B_m(\tau)$, $C(\tau)$ depend on the specific structure of the drifts, variances, and covariances of the interest rate and the factors. We denote the solutions to the differential equations governing these functions specifically as $A^*(\theta; \tau, \mathbf{b}, \mathbf{d})$, $B_1^*(\theta; \tau, \mathbf{b}, \mathbf{d}), \ldots, B_m^*(\theta; \tau, \mathbf{b}, \mathbf{d})$,

$C^*(\theta; \tau, \mathbf{b}, \mathbf{d})$, respectively, where θ denotes the vector of parameters governing the stochastic processes, and $\mathbf{b} = [a, b_1, \ldots, b_m, c]$ denotes the vector of constants that make up the boundary conditions in Equation (5).

Remark 1. The main result of this article is that the prices of a wide range of common fixed-income derivatives can be characterized solely in terms of the functions $A^*(\theta; \tau, \mathbf{b}, \mathbf{d})$, $B_1^*(\theta; \tau, \mathbf{b}, \mathbf{d})$, $\ldots$, $B_m^*(\theta; \tau, \mathbf{b}, \mathbf{d})$, $C^*(\theta; \tau, \mathbf{b}, \mathbf{d})$. Therefore we will show that if the interest rate model, regardless of how complicated it is, leads to exponential bond prices, then the prices of many interest rate-dependent claims can be easily calculated in terms of these functions.

In the case of a discount bond, the holder receives a dollar at maturity, and the boundary condition for the bond can be stated as

$$P_T(0) = 1 = \exp\left[0 r_t + \sum_{i=1}^{m} 0 x_{i,t} + 0\right]. \tag{6}$$

Therefore the specific boundary conditions for each Riccati equation are all zero, that is, $\mathbf{b} = \mathbf{0}$. Hence the price of the bond is given by

$$P_t(\tau) = \exp\left[A^*(\theta; \tau, \mathbf{0}, \mathbf{d}^*) r_t + \sum_{i=1}^{m} B_i^*(\theta; \tau, \mathbf{0}, \mathbf{d}^*) x_{i,t} + C^*(\theta; \tau, \mathbf{0}, \mathbf{d}^*)\right]. \tag{7}$$

Virtually all of the term structure models developed in the literature to date begin with interest rate/factor processes that lead to bond prices of the exponential affine form given by Equation (7). Therefore our goal in this article is to derive pricing implications for derivatives written on this specific class of interest rate processes.

1.4 Example

We now present an example of the term structure model discussed in the previous sections. The example is based on a jump-diffusion model. This example will be continued and extended in subsequent sections of the article in order to illustrate the use of the theoretical results of the article and, hopefully, to make them more concrete.

Consider the risk-neutral interest process given by the dynamics

$$dr = \kappa(\theta - r)\, dt + \sigma\, dW + J_u\, dN_u(\lambda_u) - J_d\, dN_d(\lambda_d), \tag{8}$$

where κ, θ, σ, λ_u, and λ_d are constants, while J_u and J_d are exponentially distributed random variables with positive means η_u and η_d, respectively. The interest rate in this specification displays persistence as well as skewness and excess kurtosis. The one-jump version of this process was considered in Das and Foresi (1996). A two-jump model was considered in Duffie, Pan, and

Singleton (2000). The version of Equation (3) that holds for this process is given by

$$\frac{1}{2}\sigma^2 P_{rr} + \kappa(\theta - r)P_r - P_\tau + \lambda_u E[P(r + J_u) - P(r)] \\ + \lambda_d E[P(r - J_d) - P(r)] = -drP,$$

where the subscripts on $P(r)$ denote partial derivatives. The general boundary condition on $P(r)$ that leads to Equation (5) is given by

$$P(r, \tau = 0) = \exp[ar + c].$$

Under this boundary condition, the solution to the PDDE is of the form given by Equation (4):

$$P(r) = \exp[A(\tau)r + C(\tau)], \tag{9}$$

where $A(\tau)$ and $C(\tau)$ satisfy ordinary differential equations

$$\begin{aligned}\frac{dA}{d\tau} &= -\kappa A + d \\ \frac{dC}{d\tau} &= \frac{1}{2}\sigma^2 A^2 + \kappa\theta A + \lambda_u E[e^{AJ_u} - 1] + \lambda_d E[e^{-AJ_d} - 1] \\ &= \frac{1}{2}\sigma^2 A^2 + \kappa\theta A + \lambda_u\left(\frac{\eta_u A}{1 - \eta_u A}\right) - \lambda_d\left(\frac{\eta_d A}{1 + \eta_d A}\right)\end{aligned}$$

with boundary conditions $A(\tau = 0) = a$ and $C(\tau = 0) = c$. Following the convention of the previous section, we label the vector $[a, c] = \mathbf{b}$. The solutions, labeled $A^*(\tau, \mathbf{b}, d)$ and $C^*(\tau, \mathbf{b}, d)$, are given by

$$\begin{aligned}A^*(\tau, \mathbf{b}, d) &= u_1 e^{-\kappa\tau} + u_2 \\ C^*(\tau, \mathbf{b}, d) &= \frac{u_1^2\sigma^2}{4\kappa}(1 - e^{-2\kappa\tau}) + \left[\frac{u_1 u_2 \sigma^2 + \kappa\theta u_1}{\kappa}\right](1 - e^{-\kappa\tau}) \\ &\quad + \left[\frac{u_2^2\sigma^2}{2} + \kappa\theta u_2 - \lambda_u - \lambda_d\right]\tau \\ &\quad + \frac{\lambda_u}{\kappa - d\eta_u}\log\left|\frac{(1 - \eta_u u_2)e^{\kappa\tau} - \eta_u u_1}{1 - \eta_u u_1 - \eta_u u_2}\right| \\ &\quad + \frac{\lambda_d}{\kappa + d\eta_d}\log\left|\frac{(1 + \eta_d u_2)e^{\kappa\tau} + \eta_d u_1}{1 + \eta_d u_1 + \eta_d u_2}\right| + c \\ u_1 &= a - u_2 \\ u_2 &= \frac{d}{\kappa}.\end{aligned}$$

As mentioned in the previous section, in the special case that $\mathbf{b} = [0, 0]$ and $d = -1$, the solution to Equation (9) is the price of a zero-coupon bond with maturity τ. Henceforth throughout the article, we will utilize this example to illustrate the theoretical pricing relationships and numerical methods derived in the article in the hope of making these results concrete and accessible. To begin, we specify a base set of parameters and price the bond using the equation above. The results are presented in Table 1.

Increasing the upward jump frequency of the short rate causes the bond price to fall, while the opposite happens as we increase the downward jump frequency. Intuitively, as the upward jump frequency increases, the likelihood of higher future rates increases, and since the bond price is a discounted value of these rates, bond prices drop. As both upward and downward jump frequency increase, the bond price increases slightly. For example, as both jump frequencies rise from 3 to 6 jumps per annum, the bond price rises from 0.951419 to 0.951424. This is due to the convexity of bond prices with respect to the short rate. Therefore an equal magnitude upward jump in the short rate has less effect on bond prices than an equal magnitude downward jump.

In the following sections we illustrate how to utilize the term structure solutions to price different options.

1.5 Option prices

In this section we write a general equation for the pricing of European options where the option payoff may be a general function of the interest rate r. We denote the "payoff function" at time T as $f_T(\mathbf{r}, \mathbf{x}, \hat{\tau})$, where $\mathbf{r}$ represents the sample path of interest rates and state variables, $\mathbf{x}$, up to time T. In addition, $\hat{\tau}$ is a "terminal time period" parameter, which allows the payoff function to depend on a period of time of length $\hat{\tau}$ beyond time T. Therefore the payoff of the option on its expiration date can be expressed as

$$F_T(0; \hat{\tau}) = \max[f_T(\mathbf{r}, \mathbf{x}, \hat{\tau}) - K, 0],$$

Table 1
Bond prices in the two-jump model

k	θ	σ	λ_u	λ_d	η_u	η_d	τ	r
0.2	0.1	0.1	5	5	0.005	0.005	0.5	0.1

		λ_d			
		3	6	9	12
λ_u	3	0.9514	0.9531	0.9549	0.9566
	6	0.9497	0.9514	0.9532	0.9549
	9	0.9480	0.9497	0.9514	0.9532
	12	0.9463	0.9480	0.9497	0.9514

This table presents bond prices when there are two jumps. The parameters are presented, followed by the prices for varying jump intensities, λ_u and λ_d. The jump intensities for both jumps are set at 5 jumps a year. The jumps are symmetric, in that the mean jump size for the positive and negative jumps is 50 basis points. The discount bond price is computed to be 0.9514. We then show bond prices as we vary the jump intensities from 3 to 12 jumps per annum.

where we use the notation $F_t(\tau;\hat{\tau})$ to represent the price of an option at time t with a period of time τ to expiration written on an underlying security with remaining maturity $\hat{\tau}$. As a result $T = t+\tau$. We can use the Feynman–Kac relation[9] to write the price of the option as the discounted expected value of the terminal payoff,

$$F_t(\tau;\hat{\tau}) = E_t\big[e^{-\int_t^T r_v\,dv}\max[f_T(\mathbf{r},\mathbf{x},\hat{\tau})-K,0]\big], \tag{10}$$

where the expectation is taken under the risk-neutral measure.[10] To simplify notation, we introduce the variable Z_t, defined as

$$Z_t(\tau) = \int_t^T r_v\,dv.$$

We now decompose the price of the option into two expressions as follows:

$$\begin{aligned} F_t(\tau;\hat{\tau}) &= E_t\big[e^{-Z_t} f_T(\mathbf{r},\mathbf{x},\hat{\tau})1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau})\geq K\}}\big] - E_t\big[e^{-Z_t}K1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau})\geq K\}}\big] \\ &= E_t\big[e^{-Z_t} f_T(\mathbf{r},\mathbf{x},\hat{\tau})\big]E_t\left[\frac{e^{-Z_t} f_T(\mathbf{r},\mathbf{x},\hat{\tau})1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau})\geq K\}}}{E_t\big[e^{-Z_t} f_T(\mathbf{r},\mathbf{x},\hat{\tau})\big]}\right] \qquad (11) \\ &\quad -KE_t[e^{-Z_t}]E_t\left[\frac{e^{-Z_t}1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau})\geq K\}}}{E_t[e^{-Z_t}]}\right], \qquad (12) \end{aligned}$$

where $1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau})\geq K\}}$ is an indicator function for when the option finishes up in the money. However, $E_t[e^{-Z_t}]$ is the price of a discount bond that matures at time $T\equiv t+\tau$. So, $E_t[e^{-Z_t}] = P_t(\tau)$. We define $\Pi_{0,t} = E_t[e^{-Z_t}f_T(\mathbf{r},\mathbf{x},\hat{\tau})]$, which is the present value of the underlying function that determines the payoff. Now we can rewrite the expression above as

$$\begin{aligned} F_t(\tau;\hat{\tau}) &= \Pi_{0,t}E_t\left[\frac{e^{-Z_t} f_T(\mathbf{r},\mathbf{x},\hat{\tau})1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau})\geq K\}}}{E_t[e^{-Z_t} f_T(\mathbf{r},\mathbf{x},\hat{\tau})]}\right] \\ &\qquad - KP_t(T)E_t\left[\frac{e^{-Z_t}1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau})\geq K\}}}{E_t[e^{-Z_t}]}\right]. \end{aligned}$$

It is clear that $E_t[\frac{e^{-Z_t} f_T(\mathbf{r},\mathbf{x},\hat{\tau})1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau})\geq K\}}}{E_t[e^{-Z_t} f_T(\mathbf{r},\mathbf{x},\hat{\tau})]}]$ and $E_t[\frac{e^{-Z_t}1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau})\geq K\}}}{E_t[e^{-Z_t}]}]$ are probabilities. For convenience, we denote these two probabilities as $\Pi_{1,t}$ and $\Pi_{2,t}$, respectively. The price of the option is restated as

$$F_t(\tau;\hat{\tau}) = \Pi_{0,t}\Pi_{1,t} - KP_t(\tau)\Pi_{2,t}. \tag{13}$$

The task at hand is to evaluate $\Pi_{0,t}$ and the two probabilities $\Pi_{1,t}$ and $\Pi_{2,t}$. The specific equations for $\Pi_{1,t}$ and $\Pi_{2,t}$ may be calculated solely

[9] See Duffie (1996) for a exposition of the use of these methods.

[10] All expectations from this point onward are under the risk-neutral measure unless indicated otherwise.

in terms of the functions $A^*(\theta;\tau,\mathbf{b},\mathbf{d})$, $B_1^*(\theta;\tau,\mathbf{b},\mathbf{d})$, $\ldots$, $B_m^*(\theta;\tau,\mathbf{b},\mathbf{d})$, $C^*(\theta;\tau,\mathbf{b},\mathbf{d})$ through the application of the Feynman–Kac theorem, which essentially allows us to solve for any expression of the form in Equation (10) by restating the expression as a solution to a PDDE.

In the next several sections we utilize the Feynman–Kac theorem to tackle three different types of terminal payoff functions for the pricing of interest rate derivatives:

1. Payoffs that are *linear* functions of the state variables. These may be used to price caps, floors, yield options, and slope options.
2. Payoffs that are *exponential* in the state variables, used to price bond options, forwards, and futures options.
3. Payoffs that are *integrals* of the state variables, as in the case of average rate options on the short rate and Asian options on yields.

We now examine each one in turn.

2. Option Pricing for Linear Payoffs

In this section we consider the case when the payoff function is given by a linear function of the interest rate and state variables:

$$f_T(\mathbf{r},\mathbf{x},\hat{\tau}) = k_0 r_T + k_1 x_{1,T} + \cdots + k_m x_{m,T} + k_{m+1}, \tag{14}$$

where $k_0,\ldots,k_{m+1}$ are constants. As indicated by Equation (13), the price of a European call option for this payoff function is given by

$$F_t(\tau;\hat{\tau}) = \Pi_{0,t}\Pi_{1,t} - KP_t(\tau)\Pi_{2,t}. \tag{15}$$

We now derive the function $\Pi_{0,t}$ and the two probabilities $\Pi_{1,t}$ and $\Pi_{2,t}$ for the linear payoff function. Substituting these solutions into Equation (15) will then yield the general option pricing formula for a linear terminal payoff function.

Proposition 4. (A) *The solution for* $\Pi_{0,t}$ *is given by*

$$\Pi_{0,t} = \left\{\Gamma_{0,t}\left[\frac{\partial A^*(\theta;\tau,\mathbf{b}_0,\mathbf{d}^*)}{\partial\phi} r_t + \sum_{i=1}^{m}\frac{\partial B_i^*(\theta;\tau,\mathbf{b}_0,\mathbf{d}^*)}{\partial\phi} x_{i,t} + \frac{\partial C^*(\theta;\tau,\mathbf{b}_0,\mathbf{d}^*)}{\partial\phi}\right]\right\}_{\phi=0}$$

where

$$\Gamma_{0,t} = \exp\left[A^*(\theta;\tau,\mathbf{b}_0,\mathbf{d}^*)r_t + \sum_{i=1}^{m} B_i^*(\theta;\tau,\mathbf{b}_0,\mathbf{d}^*)x_{i,t} + C^*(\theta;\tau,\mathbf{b}_0,\mathbf{d}^*)\right]$$

$$\mathbf{b}_0 = \begin{bmatrix} \phi k_0 \\ \phi k_1 \\ \vdots \\ \phi k_m \\ \phi k_{m+1} \end{bmatrix}.$$

(B) *The characteristic function for* $\Pi_{1,t}$ *is given by*

$$\widetilde{\Pi}_{1,t} = \frac{1}{\Pi_{0,t}}\left\{\frac{1}{i}\Gamma_{1,t}\left[\frac{\partial A^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*)}{\partial\omega}r_t + \sum_{i=1}^{m}\frac{\partial B_i^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*)}{\partial\omega}x_{i,t} + \frac{\partial C^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*)}{\partial\omega}\right]\right\}$$

where

$$\Gamma_{1,t} = \exp\left[A^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*)r_t + \sum_{i=1}^{m} B_i^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*)x_{i,t} + C^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*)\right]$$

$$\mathbf{b}_1 = \begin{bmatrix} i\omega k_0 \\ i\omega k_1 \\ \vdots \\ i\omega k_m \\ i\omega k_{m+1} \end{bmatrix}.$$

(C) *The characteristic function for* $\Pi_{2,t}$ *is given by*

$$\widetilde{\Pi}_{2,t} = \frac{1}{P_t(\tau)}\Gamma_{1,t}.$$

(D) *The characteristic functions in* (B) *and* (C), $\widetilde{\Pi}_{k,t}$, $k = 1, 2$, *may be inverted to obtain the probabilities* $\Pi_{k,t}$ *using a version of Fourier's theorem* (*stated earlier*):[11]

$$\Pi_{k,t} = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{1}{i\omega}e^{-i\omega K}\widetilde{\Pi}_{k,t}\right)d\omega, \quad k = 1, 2$$

Proof. See the appendix.

We now price specific options that fall into this category of payoffs.

[11] Fourier's inversion theorem for distribution functions can be found in Kendall, Ord, and Stuart (1987) and Shephard (1991)

2.1 Interest rate caps and floors, and exotics

An interest rate cap is an option that pays off when the terminal interest rate exceeds the strike K.[12] These options are one of the most widely traded instruments in the fixed-income derivatives markets. Many uses are envisaged. (i) They are routinely used by corporations to cap their funding costs. (ii) Money management companies use floors to ensure a base level of return in their portfolios. (iii) Caps and floors bear an equivalence to swaps, which also makes them useful in managing swaps portfolios. (iv) A collar is a position containing a long cap and short floor, and one popular version of these contracts is a zero-cost collar. For example, investors with a view that interest rates will rise will buy a cap and subsidize themselves by selling a floor.

While the plain vanilla form of the interest-rate cap is now widepsread in usage, more exotic options are being traded, to which the technology of this section may be put to full use. Examples are as follows: (i) Options on credit spreads are now popular, and the modeling of the term structure of spreads lends itself easily to the pricing of derivatives. (ii) With the introduction of inflation-indexed bonds, options on inflation may be valued easily, since the term structure of inflation rates is now becoming available. These options may trade off the TIPS (Treasury inflation-protection securities) market or REALs (an older OTC version of the same security). (iii) An even more exotic application is one where options on volatility levels may be priced, provided a means of ascertaining volatility is available. Implied volatilities are now readily available, and term structures of volatility are also routinely developed, making this an envisageable product. (iv) Finally, these techniques are also useful for the commodities markets in the pricing of options on convenience yields. Convenience yields are actively traded, and hedges against backwardation and contango risk may be easily set up using options on the term structure of convenience yields.

A short rate cap may also be viewed as an average rate option where the averaging period is the last instant before the contract expires. If the cap contract matures at time T, its payoff is given by

$$C_T = (r_T - K)1_{r_T \geq K}. \tag{16}$$

From Equation (16) we have the following pricing result for an interest rate cap:

$$C_0 = E_t^Q[e^{-Z_T}(r_T - K)1_{r_T \geq K}]. \tag{17}$$

This option is easily priced using the formula in Proposition 4 by setting the constants $k_0 > 0$ and $k_1 = k_2 = \cdots = k_{m+1} = 0$.

[12] This is but one possible specification of the cap, making it different from a standard option on a zero coupon bond. In another popular market convention, a cap is an option on which the payoff is based on the interest rate at option maturity, but the payment takes place at the end of the period for which the underlying interest rate applies. We take this up later in this section of Interest, the mathematical treatment for these two conventions for caps is quite different.

2.2 Yield caps and floors

We also consider the case when the cap payoff is made based on yields for an underlying period. We denote this period δ. We price a cap maturing at time T where the applicable interest rate (denoted R) is based on compounding over period length δ. The payoff at time $T+\delta$ is given by

$$\delta \max[0, R-K],$$

which translates into an equivalent payoff at time T of

$$P(r_T, T, T+\delta)\delta \max[0, R-K], \tag{18}$$

where $P(r_T, T, T+\delta)$ is the price of the bond with remaining maturity δ, denoted as $P(\delta)$ to simplify the notation. Noting that $1+R\delta = P(\delta)^{-1}$, we have

$$R = \left(\frac{1}{P(\delta)} - 1\right)\frac{1}{\delta}.$$

Using Equation (18) we may write the value of the cap at time 0 as follows:

$$\begin{aligned}
\text{cap}_{t=0} &= \delta E_0[e^{-Z_T} \max\{0, P(\delta)R - P(\delta)K\}] \\
&= \delta E_0\left[e^{-Z_T} \max\left\{0, P(\delta)\left(\frac{1}{P(\delta)} - 1\right)\frac{1}{\delta} - P(\delta)K\right\}\right] \\
&= E_0\big[e^{-Z_T} \max\{0, 1 - P(\delta) - \delta P(\delta)K\}\big] \\
&= E_0[e^{-Z_T} \max\{0, 1 - P(\delta)(1+\delta K)\}] \\
&= (1+\delta K)E_0\left[e^{-Z_T} \max\left\{0, \frac{1}{1+\delta K} - P(\delta)\right\}\right],
\end{aligned}$$

which is straightforward to value since the expression above embeds the formula for a put option of maturity T, on a zero-coupon bond of maturity $(T+\delta)$, with a strike price of $1/(1+\delta K)$. The formula for these options is developed in the following sections.

Yield options may be priced more generally by choosing the weights appropriately in the linear payoff function to match the coefficients of the yield equation. In order to price a cap on δ-maturity yield, Proposition 4 applies with $k_0 = A^*(\theta; \delta, \mathbf{0}, \mathbf{d}^*)$, $k_1 = B_1^*(\theta; \delta, \mathbf{0}, \mathbf{d}^*)$, $\ldots$, $k_m = B_m^*(\theta; \delta, \mathbf{0}, \mathbf{d}^*)$, $k_{m+1} = C^*(\theta; \delta, \mathbf{0}, \mathbf{d}^*)$, and recall that $\mathbf{d}^* = [-1, \mathbf{0}, 0]$. This version of the model may be used for caps and floors on Libor yields.

2.3 Yield combo options

A "combo" option is one where the payoff depends on a basket of yields, weighted in any chosen proportions. If the payoff is determined based on n yields with weights $x_i, i = 1..n$ and maturities $\delta_i, i = 1..n$, then the option is priced using the result in Proposition 4 with the constants set as follows: $k_0 = \sum_{i=1}^{n} x_i A^*(\theta; \delta_i, \mathbf{0}, \mathbf{d}^*)$, $k_1 = \sum_{i=1}^{n} x_i B_1^*(\theta; \delta_i, \mathbf{0}, \mathbf{d}^*)$, $\ldots$, $k_m = \sum_{i=1}^{n} x_i B_m^*(\theta; \delta_i, \mathbf{0}, \mathbf{d}^*)$, $k_{m+1} = \sum_{i=1}^{n} x_i C^*(\theta; \delta_i, \mathbf{0}, \mathbf{d}^*)$.

There are many types of combo options in the market. (i) A special case of combo options are yield curve "slope" options, based on the difference of two yields [see Duffie, Pan, and Singleton (2000)]. (ii) Differences in the levels of term structures in different markets may be exploited in these models. For example, "diff swaps" have been in place for quite a while—yield combo options are another way to achieve the benefits of diff swaps using packages of options. These "basis rate" transactions are gaining in popularity as markets across the world develop much tighter interactions and linkages. (iii) In the foreign currency markets, we have "currency coupon swaps" which are options on two different LIBOR rates. These transactions have become popular with the onset of the European Monetary System. (iv) "Basket" yield options allow trading on a basket of different interest rates, usually reducing corporation hedging costs.[13]

2.4 Example

Under the parameters of the two-jump example of the previous section, we price a cap on the short rate, at an exercise level of 10%. Using the equations from Proposition 4, we present the formula as:

(A) The solution for $\Pi_{0,t}$ is given by ($\mathbf{b}_0 = [\phi, 0], \mathbf{d}^* = [-1, 0]$)

$$\Pi_{0,t} = \left\{ \Gamma_{0,t} \left[\frac{\partial A^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*)}{\partial \phi} r_t + \frac{\partial C^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*)}{\partial \phi} \right] \right\}_{\phi=0}$$

$$\Gamma_{0,t} = \exp[A^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*) r_t + C^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*)].$$

(B) The characteristic function for $\Pi_{1,t}$ is given by ($b_1 = [i\omega, 0]$)

$$\widetilde{\Pi}_{1,t} = \frac{1}{\Pi_{0,t}} \left\{ \frac{1}{i} \Gamma_{1,t} \left[\frac{\partial A^*(\theta; \tau, \mathbf{b}_1, \mathbf{d}^*)}{\partial \omega} r_t + \frac{\partial C^*(\theta; \tau, \mathbf{b}_1, \mathbf{d}^*)}{\partial \omega} \right] \right\}$$

$$\Gamma_{1,t} = \exp[A^*(\theta; \tau, \mathbf{b}_1, \mathbf{d}^*) r_t + C^*(\theta; \tau, \mathbf{b}_1, \mathbf{d}^*)].$$

(C) The characteristic function for $\Pi_{2,t}$ is given by

$$\widetilde{\Pi}_{2,t} = \frac{1}{P_t(\tau)} \Gamma_{1,t}.$$

For illustration, we compute prices for caps given a range of jump intensities, and the results are summarized in Table 2. The value of the option for the base case ($\lambda_u = \lambda_d = 5$) is 0.0810. As one would expect, an increase in

[13] It is fortuitous that so many options may be priced as linear combinations of yields. The simplicity of these techniques is obviated, however, when nonlinear combinations of yields need to be considered or for nonlinear combinations of bond prices.

Table 2
Cap prices in the two-jump model

		λ_d			
		3	6	9	12
λ_u	3	0.0799	0.0774	0.0749	0.0724
	6	0.0837	0.0812	0.0787	0.0761
	9	0.0874	0.0849	0.0824	0.0799
	12	0.0911	0.0886	0.0861	0.0836

This table presents cap prices when there are two jumps. Prices are given for a range of jump intensities, and the value of the option for the base case ($\lambda_u = \lambda_d = 5$) is 0.0810. The parameters are the same as those in Table 1.

the downward jump frequency causes the option price to drop because the probability of the option ending in the money decreases. The opposite occurs as the upward jump frequency increases. When both upward and downward jump frequencies increase, option prices increase due to the increase in overall volatility.

3. Option Pricing for Exponential Linear Payoffs

In this section we consider the case when the payoff function is given by an exponential linear function of the interest rate and state variables:

$$f_T(\mathbf{r}, \mathbf{x}, \hat{\tau}) = \exp(k_0 r_T + k_1 x_{1,T} + \cdots + k_m x_{m,T} + k_{m+1}), \tag{19}$$

where $k_0, \ldots, k_{m+1}$ are constants. As indicated by Equation (13), the price of a European call option for this payoff function is given by

$$F_t(\tau; \hat{\tau}) = \Pi_{0,t}\Pi_{1,t} - KP_t(\tau)\Pi_{2,t}. \tag{20}$$

The following proposition develops the required option pricing formula:

Proposition 5. (A) *The solution for* $\Pi_{0,t}$ *is given by*

$$\Pi_{0,t} = \exp\left[A^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*)r_t + \sum_{i=1}^{m} B_i^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*)x_{i,t} + C^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*)\right],$$

where

$$\mathbf{b}_0 = \begin{bmatrix} k_0 \\ k_1 \\ \vdots \\ k_m \\ k_{m+1} \end{bmatrix}.$$

(B) *The characteristic function for* $\Pi_{1,t}$ *is given by*

$$\widetilde{\Pi}_{1,t} = \frac{1}{\Pi_{0,t}} \exp\Bigg[A^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*) r_t + \sum_{i=1}^{m} B_i^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*) x_{i,t} + C^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*) \Bigg], \tag{21}$$

where

$$\mathbf{b}_1 = \begin{bmatrix} (1+i\omega)k_0 \\ (1+i\omega)k_1 \\ \vdots \\ (1+i\omega)k_m \end{bmatrix}.$$

(C) *The characteristic function for* $\Pi_{2,t}$ *is given by*

$$\widetilde{\Pi}_{2,t} = \frac{1}{P_t(\tau)} \exp\Bigg[A^*(\theta;\tau,\mathbf{b}_2,\mathbf{d}^*) r_t + \sum_{i=1}^{m} B_i^*(\theta;\tau,\mathbf{b}_2,\mathbf{d}^*) x_{i,t} + C^*(\theta;\tau,\mathbf{b}_2,\mathbf{d}^*) \Bigg], \tag{22}$$

where

$$\mathbf{b}_2 = \begin{bmatrix} i\omega k_0 \\ i\omega k_1 \\ \vdots \\ i\omega k_m \end{bmatrix}.$$

(D) *We invert* $\widetilde{\Pi}_{k,t}$ *to obtain the probability* $\Pi_{k,t}$ *using a version of Fourier's theorem*:

$$\Pi_{k,t} = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left(\frac{1}{i\omega} e^{-i\omega K} \widetilde{\Pi}_{k,t} \right) d\omega, \quad k = 1, 2.$$

Proof. See the appendix.

The following three sections consider specific cases of this class of payoff function.

3.1 Discount bond options

Bond options have been traded since the late 1970s and are the oldest form of interest rate option contract.[14] A European call option on a discount bond at strike K is the right but not the obligation to purchase a discount bond with remaining maturity $\hat{\tau}$ on the expiration date of the option. The option payoff is

$$F_T(0; \hat{\tau}) = \max[P_T(\hat{\tau}) - K, 0].$$

The price of the bond option is the discounted expected value of the terminal payoff:

$$\begin{aligned} F_t(\tau; \hat{\tau}) &= E_t\left[e^{-\int_t^T r_v dv} \max[P_T(\hat{\tau}) - K, 0]\right] \\ &= E_t\left[e^{-Z_t(\tau)} P_T(\hat{\tau})\right]\Pi_{1,t} - KP_t(\tau)\Pi_{2,t} \\ &= P_t(\tau + \hat{\tau})\Pi_{1,t} - KP_t(\tau)\Pi_{2,t}. \end{aligned} \tag{23}$$

This is easily priced, an examination of the results of Proposition 5 reveals that setting the constants to the following values provides the value of the bond option: $k_0 = A^*(\theta; \delta, \mathbf{0}, \mathbf{d}^*)$, $k_1 = B_1^*(\theta; \delta, \mathbf{0}, \mathbf{d}^*), \ldots, k_m = B_m^*(\theta; \delta, \mathbf{0}, \mathbf{d}^*)$, $k_{m+1} = C^*(\theta; \delta, \mathbf{0}, \mathbf{d}^*)$.

3.2 Futures and forwards on discount bonds

We begin with a derivation of forward and futures prices. Let $F_{d,t}(\tau; \hat{\tau})$ denote the τ-period-ahead forward price of a $\hat{\tau}$-period bond at time t. By definition, the forward price is simply the ratio of the $(\tau + \hat{\tau})$-period bond price over the τ-period bond price:

$$F_{d,t}(\tau; \hat{\tau}) = \frac{P_t(\tau + \hat{\tau})}{P_t(\tau)}. \tag{24}$$

Let $F_{u,t}(\tau; \hat{\tau})$ denote the τ-period-ahead futures price of a $\hat{\tau}$-period bond at time t. The futures price is given by a simple application of the exponential model:

$$\begin{aligned} F_{u,t}(\tau; \hat{\tau}) = \exp\Bigg[& A^*(\theta; t+\tau, \mathbf{b}, \mathbf{d}) r_t + \sum_{i=1}^{m} B_i^*(\theta; t+\tau, \mathbf{b}, \mathbf{d}) x_{i,t} \\ & + C^*(\theta; t+\tau, \mathbf{b}, \mathbf{d}) \Bigg], \end{aligned} \tag{25}$$

[14] Pricing formulas for bond options were available as early as that of the Vasicek (1977) model and the Cox, Ingersoll, and Ross (1985) model. Since then, many other articles have dealt with bond option models: Courtadon (1982), Ho and Lee (1986), Jamshidian (1989), Carverhill and Clewlow (1990), Shirakawa (1991), Heath, Jarrow, and Morton (1992), Longstaff and Schwartz (1992), Geman and Yor (1993), Heston (1993), Naik and Lee (1993), Yor (1993), Eydeland and Geman (1994), Das and Foresi (1996), Das (1997), Duffie, Pan, and Singleton (2000), Leblanc and Scaillet (1998), and Heston and Nandi (1999).

where

$$\mathbf{b} = \begin{bmatrix} A^*(\theta; t+\tau, \mathbf{0}, \mathbf{d}^*) \\ B_1^*(\theta; t+\tau, \mathbf{0}, \mathbf{d}^*) \\ \vdots \\ B_m^*(\theta; t+\tau, \mathbf{0}, \mathbf{d}^*) \\ C^*(\theta; t+\tau, \mathbf{0}, \mathbf{d}^*) \end{bmatrix}$$
$$\mathbf{d} = [0, \mathbf{0}, 0].$$

Note here that $d \neq d^*$ as in the previous subsection.

3.3 Discount bond futures options

Futures options are traded on exchanges and are typically liquid contracts. A European call option on a discount bond future at strike K is the right but not the obligation to purchase a bond future with remaining maturity $\hat{\tau}$ on the expiration date of the option. Let $F_t(\tau; \hat{\tau}, \tau')$ be the price of an option with time-to-expiration τ, written on a futures contract with time to maturity $\hat{\tau}$ that calls for the delivery of a discount bond with time to maturity τ'. The option payoff is

$$F_T(0; \hat{\tau}, \tau') = \max[F_{u,T}(\hat{\tau}; \tau') - K, 0].$$

The futures option is easily priced using the results of Proposition 5. Setting the constants to the following values provides the value of the bond option: $k_0 = A^*(\theta; \delta, \mathbf{0}, \mathbf{d})$, $k_1 = B_1^*(\theta; \delta, \mathbf{0}, \mathbf{d}), \ldots,$ $k_m = B_m^*(\theta; \delta, \mathbf{0}, \mathbf{d})$, $k_{m+1} = C^*(\theta; \delta, \mathbf{0}, \mathbf{d})$, where $\mathbf{d} = [0, \mathbf{0}, 0]$. A common type of contract to which these techniques apply are Euro-currency futures options.

3.4 Example

As an example of the models in this section, we price bond options on discount bonds of half-year remaining maturity ($\hat{\tau} = 0.5$), where the option maturity is also a half year ($\tau = 0.5$). The representative equations from Proposition 5 are as follows:

(A) The solution for $\Pi_{0,t}$ is given by ($\mathbf{b}_0 = [A^*(\theta; \hat{\tau}, \mathbf{b}_0, \mathbf{d}^*), C^*(\theta; \hat{\tau}, \mathbf{b}_0, \mathbf{d}^*)]$, $\mathbf{d}^* = [-1, 0]$)

$$\Pi_{0,t} = \exp[A^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*) r_t + C^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*)].$$

(B) The characteristic function for $\Pi_{1,t}$ is given by

$$\widetilde{\Pi}_{1,t} = \frac{1}{\Pi_{0,t}} \exp[A^*(\theta; \tau, \mathbf{b}_1, \mathbf{d}^*) r_t + C^*(\theta; \tau, \mathbf{b}_1, \mathbf{d}^*)]$$

Table 3
Bond option pricing in the two-jump model

		λ_d			
		3	6	9	12
λ_u	3	0.0362	0.0390	0.0423	0.0462
	6	0.0335	0.0357	0.0385	0.0419
	9	0.0313	0.0330	0.0353	0.0381
	12	0.0297	0.0309	0.0326	0.0348

This table presents prices for bond options on discount bonds of half-year remaining maturity ($\hat{\tau} = 0.5$), where the option maturity is also a half year ($\tau = 0.5$). Prices are computed for options given a range of jump intensities, and the results are summarized in Table 3. The value of the option for the base case ($\lambda_u = \lambda_d = 5$) is 0.0361.

$$\mathbf{b}_1 = \begin{bmatrix} (1+i\omega)A^*(\theta;\tau,\mathbf{b}_0,\mathbf{d}^*) \\ (1+i\omega)C^*(\theta;\hat{\tau},\mathbf{b}_0,\mathbf{d}^*) \end{bmatrix}.$$

(C) The characteristic function for $\Pi_{2,t}$ is given by

$$\widetilde{\Pi}_{2,t} = \frac{1}{P_t(\tau)} \exp[A^*(\theta;\tau,\mathbf{b}_2,\mathbf{d}^*)r_t + C^*(\theta;\tau,\mathbf{b}_2,\mathbf{d}^*)]$$

$$\mathbf{b}_2 = \begin{bmatrix} i\omega A^*(\theta;\hat{\tau},\mathbf{b}_0,\mathbf{d}^*) \\ i\omega C^*(\theta;\hat{\tau},\mathbf{b}_0,\mathbf{d}^*) \end{bmatrix}.$$

For illustration, we compute prices for options given a range of jump intensities, and the results are summarized in Table 3. The value of the option for the base case ($\lambda_u = \lambda_d = 5$) is 0.0361. As one would expect, an increase in the downward jump frequency causes option prices to increase. This is because an increase in the downward jump frequency in the short rate translates into an increase in the *upward* jump frequency of bond prices. Therefore bonds of all maturities have a higher probability of being in the money. The opposite occurs with an increase in upward jump frequency.

4. Option Pricing for Integro-Linear Payoffs

This section considers the case when the payoff function is given by a path integral of a linear function of the interest rate and state variables:

$$f_T(\mathbf{r},\mathbf{x},\hat{\tau}) = \int_t^T (k_0 r_v + k_1 x_{1,v} + \cdots + k_m x_{m,v} + k_{m+1})\,dv, \tag{26}$$

where $k_0,\ldots,k_{m+1}$ are constants. As indicated by Equation (13), the price of a European call option for this payoff function is given by

$$F_t(\tau;\hat{\tau}) = \Pi_{0,t}\Pi_{1,t} - KP_t(\tau)\Pi_{2,t}. \tag{27}$$

Proposition 6. (A) *The solution for* $\Pi_{0,t}$ *is given by*

$$\Pi_{0,t} = \left\{\Phi_t \times \left[\frac{\partial A^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)}{\partial\phi} r_t + \sum_{i=1}^{m}\frac{\partial B_i^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)}{\partial\phi} x_{i,t} + \frac{\partial C^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)}{\partial\phi}\right]\right\}_{\phi=0},$$

where

$$\Phi_t = \exp\left[A^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)r_t + \sum_{i=1}^{m} B_i^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)x_{i,t} + C^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)\right]$$

$$\mathbf{d}_0' = \begin{bmatrix}\phi k_0 - 1\\ \phi k_1\\ \vdots\\ \phi k_m\\ \phi k_{m+1}\end{bmatrix}.$$

(B) *The characteristic function for* $\Pi_{1,t}$ *is given by*

$$\widetilde{\Pi}_{1,t} = \frac{1}{\Pi_{0,t}}\left\{\Phi_t \times \left[\frac{\partial A^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)}{\partial\phi} r_t + \sum_{i=1}^{m}\frac{\partial B_i^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)}{\partial\phi} x_{i,t} + \frac{\partial C^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)}{\partial\phi}\right]\right\}_{\phi=i\omega}.$$

(C) *The characteristic function for* $\Pi_{2,t}$ *is given by*

$$\widetilde{\Pi}_{2,t} = \frac{1}{P_t(\tau)}\exp\left[A^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)r_t + \sum_{i=1}^{m} B_i^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)x_{i,t} + C^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)\right]_{\phi=i\omega}.$$

(D) *We invert* $\widetilde{\Pi}_{k,t}$ *to obtain the probability* $\Pi_{k,t}$ *using a version of Fourier's theorem*:

$$\Pi_{k,t} = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{1}{i\omega}e^{-i\omega K}\widetilde{\Pi}_{k,t}\right)d\omega, \;\; k = 1, 2.$$

Proof. See the appendix.

This payoff class relates to the pricing of average (Asian) options on short rates and yields.

4.1 Asian options

Asian options[15] have several uses: (i) Banks and corporations use them to hedge their financing costs over an extended period of time rather than rely on more traditional contracts such as caps, floors, and collars. (ii) Corporations that have cash flows over a period of time may use an Asian option instead of a series of conventional options to hedge the risks associated with these cash flows. Asian options are often cheaper than regular options, which makes hedging cost effective. (iii) The writers of caps and floors may use Asian options to hedge their risk on these contracts over several maturities. (iv) Interest differentials are known to follow mean-reverting processes, and Asian options written on the average interest differential of two currencies may be used to hedge risk in a portfolio of long-term foreign currency options over a range of maturities. (v) Binary Asian options may be used to cover "event risk", such contracts pay off a fixed amount only if an event occurs. An example of such contracts is one where two parties contract on whether market convergence between two interest rates will occur or not. In this setting, the rationale for the binary Asian option lies in the fact that interest rates will be in one of two regimes (high or low) depending on the outcome of convergence. Since regimes are often difficult to detect empirically, writing options on the average of a financial variable over a period of time is more likely to ensure that a financial variable actually resides within a regime than when a variable is examined only at one point in time. (vi) Asian options are less susceptible to market manipulation by the option's counterparties, since it is harder to manipulate a market over an extended period of time. (vii) And finally, Fed funds futures and options are contingent on the average Fed funds rate during a month.[16]

Proposition 6 provides the pricing of Asian options on the short rate and yields. This complements the work of Longstaff (1995), who has developed a similar single-factor model using different methods. Geman and Yor (1993) use Bessel process methods to value perpetuities in both the O-U and square-root process models.[17] In this article, alternative methods for *finite time* integrals of mean-reverting Brownian motions are developed by means of state-space expansion.[18]

[15] In the literature on Asian options, various analytical solutions have been obtained. Geman and Yor (1993) provide a solution for the arithmetic average option when the underlying follows a Bessel process. Most of the work done on techniques for pricing Asian options focuses on numerical techniques such as Monte Carlo simulation or lattice-based methods. Examples of interesting numerical techniques for the Asian option problem with geometric Brownian motions include Yor (1993), De-Schepper, Teunen, and Goovaerts (1994), and Dewynne and Wilmott (1995) Barraquand and Pudet (1996). In addition, the overwhelming majority of work has focused on Asian options written on a stock price or a foreign exchange rate, where the use of geometric Brownian motion may be deemed appropriate.

[16] We are grateful to the referee for suggestions of possible contracts that may be priced using these techniques.

[17] Perpetuities are also integrals of exponentials of a Brownian motion and hence are logically subsumed within the framework of Geman and Yor (1993). This issue also connects with the work on perpetuities by Dufresne (1990). See also Bouaziz, Briys, and Crouhy (1994).

[18] In an earlier version of the article, our method for binary Asian options on jump diffusion processes had not been extended to standard Asian options, developed originally by Bakshi and Madan (1997, 2000),

Asian options on the short rate are priced using a special case of Proposition 6 where $k_0 > 0, k_1 = k_2 = \cdots = k_{m+1}$. However, Asian options on yields are far more widely used, such as in the case of options on the average of the 3-or 6-month yield. These are also amenable to Proposition 6 with $k_0 = A^*(\theta; \delta, \mathbf{0}, \mathbf{d}^*), k_1 = B_1^*(\theta; \delta, \mathbf{0}, \mathbf{d}^*), \ldots, k_m = B_m^*(\theta; \delta, \mathbf{0}, \mathbf{d}^*), k_{m+1} = C^*(\theta; \delta, \mathbf{0}, \mathbf{d}^*)$, where the option is written on the average of the δ-maturity yield.

4.2 Example

As an example, we extend the two-jump model to the pricing of an Asian option on the short rate. The option maturity is a half year ($\tau = 0.5$) and the exercise level of the average rate is 10%. The equations from Proposition 6 are

(A) The solution for $\Pi_{0,t}$ is given by

$$\Pi_{0,t} = \left\{ \Phi_t \times \left[\frac{\partial A^*(\theta; \tau, \mathbf{0}, \mathbf{d}_0)}{\partial \phi} r_t + \frac{\partial C^*(\theta; \tau, \mathbf{0}, \mathbf{d}_0)}{\partial \phi} \right] \right\}_{\phi=0}$$

$$\Phi_t = \exp[A^*(\theta; \tau, \mathbf{0}, \mathbf{d}_0) r_t + C^*(\theta; \tau, \mathbf{0}, \mathbf{d}_0)]$$

$$\mathbf{d}_0' = \begin{bmatrix} \phi - 1 \\ 0 \end{bmatrix}.$$

(B) The characteristic function for $\Pi_{1,t}$ is given by

$$\widetilde{\Pi}_{1,t} = \frac{1}{\Pi_{0,t}} \left\{ \Phi_t \times \left[\frac{\partial A^*(\theta; \tau, \mathbf{0}, \mathbf{d}_0)}{\partial \phi} r_t + \frac{\partial C^*(\theta; \tau, \mathbf{0}, \mathbf{d}_0)}{\partial \phi} \right] \right\}_{\phi=i\omega}.$$

(C) The characteristic function for $\Pi_{2,t}$ is given by

$$\widetilde{\Pi}_{2,t} = \frac{1}{P_t(\tau)} \exp[A^*(\theta; \tau, \mathbf{0}, \mathbf{d}_0) r_t + C^*(\theta; \tau, \mathbf{0}, \mathbf{d}_0)]_{\phi=i\omega}.$$

While the jump example is merely illustrative, numerical examples for the Asian option model and other models are provided in Section 6.

5. Model Implementation

The solutions provided in the previous sections provide a convenient set of results that should allow researchers to write down pricing solutions to exponential term structure models in one quick step. However, the implementation of these models for actual pricing purposes requires calibration of the base term structure model against a set of data. In this section we extend the approach in Duffie and Kan (1996) to show how calibration can readily be accomplished, and subsequently we show how to implement option pricing using the calibrated model.

for diffusion processes. Here we provide complete solutions to all Asian-type contracts in the particular framework of this article, which complements the results of Bakshi and Madan, and more recently, for jump diffusions, by Duffie, Pan, and Singleton (2000).

Calibration of the model using a cross-section of bond prices provides one way of obtaining the risk-neutral parameters required for derivative security pricing. In the class of models investigated in this article, it is possible to obtain parameter estimates directly off the Riccati equations for the term structure model. We call this approach "pricing by estimation of primitives."

5.1 Calibration methodology

We assume the existence of cross-sectional data on bond prices, that is, there are a set of N bonds at time t: $\{P_t(\tau_k)\}_{k=1,\ldots,N}$. Alternatively, estimation may be undertaken using a full panel dataset of T observations, in which case we have $\{P_t(\tau_k)\}_{k=1,\ldots,N,t=1,\ldots,T} \in R^{T\times N}$. These prices directly map into a set of yields: $Y_t(\tau_k)$, $\forall k, t$. The yields are given by

$$Y(\tau) = -\frac{1}{\tau}\ln[P(\tau)] = -\frac{1}{\tau}\left[A(\tau)r_t + \sum_{i=1}^{m} B_i(\tau)x_i + C(\tau)\right]. \tag{28}$$

The coefficients in the pricing equation, $A(\tau), B_1(\tau), \ldots, B_m(\tau), C(\tau)$, are solutions to the Riccati equation system. Cross-sectional calibration is possible using the closed-form solutions for $P(\tau)$ as was undertaken in Brown and Dybvig (1986).[19]

Given the set of affine processes for the term structure model and data on the state variables, starting with the initial condition, and a guess of the parameters of the stochastic processes, we use the Riccati equations to generate values of $A(\tau_k), B_1(\tau_k), \ldots, B_m(\tau_k), C(\tau_k)$ for $k = 1, \ldots, N$ via forward propogation in time. Using vectorization, this is done in one pass and results in fast and accurate computation for the entire set of bonds. These values and the data on the state variables $(r, x_1, \ldots, x_m)$ determine the right-hand side of Equation (28). Least squares minimization[20] of fitted versus actual yields allows rapid convergence of the algorithm to yield the vector of parameter estimates θ.

The algorithm may be summarized as follows:

$$\min_{\theta} \sum_{t=1}^{T}\sum_{k=1}^{N} \varepsilon_t[\theta(\tau_k)]^2$$

[19] The estimation literature for the term structure has been extended at a galloping rate. A representative sample of estimation methods is covered by the articles of Ait-Sahalia (1996), Attari (1997), Babbs and Webber (1995), Balduzzi et al. (1998), Balduzzi, Das, and Foresi (1998), Brandt and Santa-Clara (1998), Brenner, Harjes, and Kroner (1996), Chacko (1998), Chan et al. (1992), Conley et al. (1997), Dai and Singleton (2000, 2001), Das (1998), Duffie and Glyn (1996), Gray (1996), Koedijk et al. (1996), Naik and Lee (1993), Pritsker (1998), Singleton (1997), and Stanton (1997).

[20] Any other criterion may be used as well.

subject to

$$\varepsilon_t[\theta(\tau_k)] = Y(\tau_k) + \frac{1}{\tau}\left[A(\tau_k)r_t + \sum_{i=1}^{m} B_i(\tau_k)x_{it} + C(\tau_k)\right]$$

$$\begin{aligned}
\frac{\partial A}{\partial \tau} &= \frac{1}{2}\sum_{i=1}^{n}\sigma_{r,i}^2 A^2 + \mu_r A + d,\ \forall\tau_k,\ d=-1,\ A(0)=0\\
\frac{\partial B_1}{\partial \tau} &= \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}\Lambda_{x_1,ij}^2 B_i B_j + \sum_{i=1}^{m}\rho_{x_1,i}AB_i + \sum_{i=1}^{n}\sigma_{x_1,i}^2 A^2\\
&\quad + \mu_{x_1}A + \sum_{i=1}^{m}\alpha_{x_1,i}B_i, \quad \forall\tau_k,\ [B_1(0),\dots,B_m(0)]' = 0\\
&\ \ \vdots\\
\frac{\partial B_m}{\partial \tau} &= \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}\Lambda_{x_m,ij} B_i B_j + \sum_{i=1}^{m}\rho_{x_m,i}AB_i + \sum_{i=1}^{n}\sigma_{x_m,i}^2 A^2\\
&\quad + \mu_{x_m}A + \sum_{i=1}^{m}\alpha_{x_m,i}B_i\,\forall\tau_k,\ [B_1(0),\dots,B_m(0)]' = 0\\
\frac{\partial C}{\partial \tau} &= \frac{1}{2}\sum_{j=1}^{n}\sigma_j A^2 + \mu A + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}\Lambda_{ij}B_iB_j\\
&\quad + \sum_{i=1}^{m}\alpha_i B_i + \sum_{i=1}^{m}\rho_i AB_i + \sum_{i=1}^{l}\lambda_i\\
&\quad \times\left(E\left[e^{AJ_{r,i}+B_1J_{x,1i}+\cdots+B_mJ_{x,mi}}\right]-1\right) \quad \forall\tau_k,\ C(0)=0.
\end{aligned} \tag{29}$$

This approach has many useful features. First, since the Riccati equation system [Equation (29)] consists entirely of first-order ordinary differential equations, generation of the value set $A(\tau_k), B_1(\tau_k), \dots, B_m(\tau_k), C(\tau_k)$ for a given θ is very accurate.[21] Second, since the calibration equation is linear, and the objective function is quadratic, we obtain a well-behaved optimization problem. Third, we retain the choice of undertaking calibration either for the entire panel of data or for a single cross-section only. Fourth, since the information used relates directly to the prices of derivative securities, all estimated parameters are with respect to the risk-neutral measure and may be used for pricing immediately.[22]

[21] Indeed, standard mathematical packages yield highly accurate results. We found the `ode45` function in Matlab to be extremely fast and accurate. This is but one example of the power of using the original Riccati equations. Other facile implementations using characteristic function estimators are considered in Chacko (1998) and Singleton (1997).

[22] Appendix B gives an example of the implementation for the Vasicek (1997) model. Extending the estimation procedure to calibration off option prices involves only one extra dimension in the ODE generator for the parameter ω. See the following section.

5.2 The implementation of option pricing

As an example, we consider the pricing of bond options for which the equation is $F_t(\tau;\hat{\tau}) = P_t(\tau+\hat{\tau})\Pi_{1,t} - KP_t(\tau)\Pi_{2,t}$, where K is the strike price. There are four components to this model: (i) an underlying bond of maturity $(\tau+\hat{\tau})$, (ii) a bond of the same maturity (τ) as the option, (iii) the probability of the option finishing in the money $(\Pi_{2,t})$, and (iv) the present value of a dollar conditional on the option finishing in the money $(\Pi_{1,t})$. Since the first two components are directly observable from the market, we need only compute the two probability values.

Since $\Pi_{2,t}$ is not directly computable, we obtain its characteristic function, $\widetilde{\Pi}_{2,t}$, which is the solution to the Riccati differential equation system [Equation (29)]. We propogate the differential equation system forward to time T, beginning with the appropriate initial conditions, which are (see Proposition 5)

$$\mathbf{b} = \begin{bmatrix} i\omega A^*(\theta;\hat{\tau},\mathbf{0},\mathbf{d}^*) \\ i\omega B_1^*(\theta;\hat{\tau},\mathbf{0},\mathbf{d}^*) \\ \vdots \\ i\omega B_m^*(\theta;\hat{\tau},\mathbf{0},\mathbf{d}^*) \\ i\omega C^*(\theta;\hat{\tau},\mathbf{0},\mathbf{d}^*) \end{bmatrix}. \tag{30}$$

The values $A^*(\theta;\hat{\tau},\mathbf{0},\mathbf{d}^*)$, $B_1^*(\theta;\hat{\tau},\mathbf{0},\mathbf{d}^*), \ldots, B_m^*(\theta;\hat{\tau},\mathbf{0},\mathbf{d}^*)$, $C^*(\theta;\hat{\tau},\mathbf{0},\mathbf{d}^*)$ are the coefficients from boundary conditions that have been computed from the calibration step and are therefore already available. Hence the vector $\mathbf{b}$ is completely known, and forms an observable initial condition for forward propogation via the Riccati system. For implementation purposes we discretize the state space on which the Fourier inversion parameter ω resides, that is, generate a finite support set $\omega \in \{0, \omega_1, \omega_2, \ldots, \bar{\omega}\}$ with equal intervals $\Delta\omega$. This generates via the Riccati system a set of values of the probability $\widetilde{\Pi}_{2,t}(\omega)$ for each value of ω. Fourier inversion yields the probabilities $\Pi_{1,t}$ and $\Pi_{2,t}$.[23]

6. Illustrative Examples

In this section we present examples illustrating the techniques of the article. The purpose of the section is not to develop pricing solutions for new models but

[23] Since the Fourier inversion involves a complex integral from zero to infinity, it is often numerically unstable. One approach is to simply use an integration package such as that available from *Mathematica*. Otherwise a suitable discretization also leads to fairly accurate results. Since the upper limit of the integral is infinity, any numerical scheme that truncates the integral needs to check carefully that the tail of the integral has died out before the truncation point. For a similar discussion, see also the use of Fourier inversion via integration in estimation technology developed in Singleton (1997). Singleton provides an extensive discussion of the appropriate choice of discrete grid for the implementation of the procedure.

instead is to illustrate how to use the techniques developed in earlier sections of this article. This is best done in the context of simple models. We analyze exotic options such as range-Asians and credit spread calls, and we provide results for a version of the jump diffusion example that has been used throughout this article.

6.1 Range-Asian options

We explore a more exotic option, the range-Asian. The process on which this option is written is the simple Vasicek model, that is, Equation (8) devoid of the jump component. The range-Asian is an interesting option to analyze because it offers a good setting in which the joint effects of the mean rate, θ, and the rate of mean reversion, k, may be examined. In general, a range option is one that pays off a certain amount each day if the value of the underlying variable lies within a pre specified range. The range-Asian pays off each day that the current average up to that date remains within prespecified limits. These options have daily payoffs that are based on whether the average interest rate up to time t lies within a prespecified range $[a(t), b(t)]$, $a(t) < b(t)$, $\forall t \in [0, T]$. In the article we assume that $a(t) = a$ and $b(t) = b$, without loss of generality. The value of these options is simply

$$R[a(t), b(t), T] = \frac{1}{d}\sum_{j=1}^{d}[Q(a(t), t) - Q(b(t), t)]$$
$$d = \text{Flr}(T * 365)$$
$$t = \frac{j}{365},$$

where $Q(.)$ is the value of a binary Asian option, and $\text{Flr}(x)$ is a function that returns the greatest integer less than or equal to K. Our analyses utilize both the square-root and the O-U process models.

Pricing examples for range-Asian options are presented in Table 4. These prices increase when the range widens. When the mean rate θ lies inside the range, increases in mean reversion (k) drive the price upward. This is because, as k rises, the likelihood of the interest rate remaining within the range increases, thereby raising value. When the mean lies outside the range, option prices decrease when k increases because the interest rate is less likely to remain in the desired range. This is true of both cases, when the mean is above and below the range, that is, $\theta = 0.15$ and $\theta = 0.05$, respectively.

6.2 The jump diffusion model

In this section we analyze the pricing of Asian options in a jump diffusion framework. In particular, we extend the results of Das and Foresi (1996) to the pricing of Asian options on jump diffusions. The underlying interest rate process is as follows:

$$dr = k(\theta - r)\,dt + \sigma\,dz + J(\psi, \alpha)\,dQ(\lambda).$$

Table 4
Pricing range-Asian options

Range	$k = 0.5$	$k = 1.5$	$k = 2.5$
$\theta = 0.05$			
$a = 0.09, b = 0.11$	0.3718	0.3539	0.3075
$a = 0.08, b = 0.12$	0.6631	0.6461	0.5923
$a = 0.07, b = 0.13$	0.8388	0.8353	0.8066
$\theta = 0.10$			
$a = 0.09, b = 0.11$	0.3818	0.4120	0.4424
$a = 0.08, b = 0.12$	0.6741	0.7132	0.7488
$a = 0.07, b = 0.13$	0.8435	0.8714	0.8922
$\theta = 0.15$			
$a = 0.09, b = 0.11$	0.3823	0.3810	0.3406
$a = 0.08, b = 0.12$	0.6717	0.6646	0.6150
$a = 0.07, b = 0.13$	0.8374	0.8265	0.7938

The following table presents the values of the range-Asian option. This option is written over a fixed number of days. Every day the option pays off if the average interest rate up to that day lies within a range (a, b). The payoff is a dollar divided by the number of days the option is written for. The values in this table are for a range-Asian option with maturity $T = 0.2$ years, that is, 73 days. The parameters that are varied are (a) mean reversion (k), (b) lower range limit (a), and (c) upper range limit (b). The base parameters used are initial interest rate ($r_0 = 0.1$), time to maturity ($T = 0.2$), mean rate ($\theta = 0.1$), market price of risk ($\xi = 0$), volatility ($\eta = 0.2$).

Here, k is the rate of mean reversion, θ is the long-run mean of the interest rate, σ is the coefficient of diffusion volatility, and dz is the Wiener increment. The jump portion has a point process Q with jump arrival intensity λ and the jump J has a sign determined by parameter ψ, which represents the probability of a positive jump. The parameter $\alpha > 0$ defines the jump size and is the distribution parameter for an exponential distribution, such that it has mean $\frac{1}{\alpha}$, that is, the probability density function is given by $f(J) = \alpha e^{-\alpha J}$. As an example, we price bonds and options with the following base case parameters:

k	θ	σ	λ	α	ψ	r	τ
2	0.1	0.02	5	50	1	0.1	3

Bond maturity is denoted τ. This results in a discount bond price of $P(0.1, 3) = 0.6545$. As an illustration we choose one interesting case, that is, $\psi = 1$, which indicates that there will only be positive jumps, and this diminishes the probability of negative interest rates, but also injects a substantial quantity of positive skewness in interest rates. We shall vary the jump intensity (λ) from 0 to 10 to see how increasing skewness affects the price of the binary Asian option and the standard Asian option.

Pricing results are presented in Table 5. The standard Asian option of course has lower values, and since it is probability weighted, the payoff effect is always strong enough to outweigh the discounting effect, resulting in a monotonically increasing option value as skewness increases.

As λ increases, the value of the binary Asian option first rises and then declines. The intuition for this is straightforward. Increasing positive skewness forces the binary option further into the money, making it more valuable. At the same time,

Table 5
Pricing interest rate securities in the jump diffusion model

k	θ	σ	λ	α	ψ	r	τ
2	0.1	0.02	5	50	1	0.1	3

λ	Bond price	Binary Asian	Standard Asian
0	0.7409	0.3660	0.0015
1	0.7228	0.6059	0.0063
2	0.7051	0.6737	0.0116
3	0.6878	0.6802	0.0169
4	0.6710	0.6693	0.0219
5	0.6545	0.6541	0.0268
6	0.6385	0.6384	0.0313
7	0.6229	0.6229	0.0357
8	0.6076	0.6076	0.0398
9	0.5927	0.5927	0.0436
10	0.5782	0.5782	0.0473

The following results present prices of bonds, binary Asian options, and standard Asian options in the jump diffusion model. Results are presented for a range of skewness levels, that is, as λ ranges from 0 to 10. Base parameters are presented in the first tableau and results in the second.

the skewness increases the average discount rate for the payoff, reducing the value of the option. When λ reaches the value of 3, the binary option is maximized in value and declines thereafter as the discounting effect swamps the increasing payoff.

6.3 Pricing credit spread options

In this, our final example, we employ a two-factor model for pricing a call option on the credit spread. Our model is a special case of the model in Duffie, Pedersen, and Singleton (2000). We assume that the interest rate follows an affine square-root diffusion model given by

$$dr = (\mu_0 + \mu_1 r)\, dt + \eta\sqrt{r}\, dW,$$

and the spread process is described in the following stochastic differential equation:

$$ds = (\gamma_0 + \gamma_1 s)\, dt + \sigma\sqrt{s}\, dZ.$$

After applying Ito's lemma, the canonical PDE (in P) obtained is as follows:

$$\frac{1}{2}\eta^2 r\frac{\partial^2 P}{\partial r^2} + \frac{1}{2}\sigma^2 s\frac{\partial^2 P}{\partial s^2} + (\mu_0 + \mu_1 r)\frac{\partial P}{\partial r} + (\gamma_0 + \gamma_1 s)\frac{\partial P}{\partial s} - \frac{\partial P}{\partial \tau} + drP = 0.$$

We guess and verify a solution to this PDE: $P(r, \tau) = \exp[A(\tau)r + B(\tau)s + C(\tau)]$. The specific problem we wish to solve is defined by the choice of values $\{a, b, c, d\}$, where $A(0) = a, B(0) = b, C(0) = c$. Therefore $P(r, 0) = \exp[ar + bs + c]$. In the special case when $a = b = c = 0$, $d = -1$, we obtain the bond price. Solving the PDE by separation of variables, we obtain three ODEs which we solve entirely in closed form. The solutions are as follows:

$$A(\tau, a, d) = \frac{2}{\eta^2}\left(\frac{[2u_2 + a\eta^2]u_1 e^{u_1\tau} - [2u_1 + a\eta^2]u_2 e^{u_2\tau}}{[2u_1 + a\eta^2]e^{u_2\tau} - [2u_2 + a\eta^2]e^{u_1\tau}}\right)$$

$$B(\tau, b, d) = \frac{2}{\sigma^2}\left(\frac{[2v_2 + b\sigma^2]v_1 e^{v_1\tau} - [2v_1 + b\sigma^2]v_2 e^{v_2\tau}}{[2v_1 + b\sigma^2]e^{v_2\tau} - [2v_2 + b\sigma^2]e^{v_1\tau}}\right)$$

$$C(\tau, c, d) = c + \frac{2\mu_0}{\eta^2}\ln\left(\frac{2[u_1 - u_2]}{[2u_1 + a\eta^2]e^{u_2\tau} - [2u_2 + a\eta^2]e^{u_1\tau}}\right) + \frac{2\gamma_0}{\sigma^2}\ln\left(\frac{2[v_1 - v_2]}{[2v_1 + b\sigma^2]e^{v_2\tau} - [2v_2 + b\sigma^2]e^{v_1\tau}}\right)$$

$$u_1 = \frac{\mu_1 + \sqrt{\mu_1^2 - 2\,d\eta^2}}{2}$$

$$u_2 = \frac{\mu_1 - \sqrt{\mu_1^2 - 2\,d\eta^2}}{2}$$

$$v_1 = \frac{\gamma_1 + \sqrt{\gamma_1^2 - 2d\gamma^2}}{2}$$

$$v_2 = \frac{\gamma_1 - \sqrt{\gamma_1^2 - 2d\gamma^2}}{2}.$$

The spread call is given by a payoff that is based on a face value of \$100,000 and is paid off on the difference between the spread at maturity and the strike spread (K). We apply Proposition 4 to the model here. Simple inspection gives the vectors $\mathbf{b}_0 = [0, \phi, 0]$, $\mathbf{b}_1 = [0, i\omega, 0]$.

We computed option values for a range of spread levels and spread volatilities, the results are presented in Table 6.

Thus we have demonstrated with several numerical examples that the solutions derived in this article are easy to implement in practice. Since the solution

Table 6
Pricing credit spread options

μ_0	μ_1	γ_0	γ_1	η	σ	r_t	s_t	K	t
0.02	−0.2	0.015	−0.5	0.10	0.20	0.10	0.03	0.03	1.0

	s_t				
σ	0.01	0.02	0.03	0.04	0.05
0.10	74.21	232.21	492.13	843.62	1266.44
0.20	359.99	631.34	954.42	1317.45	1714.26
0.30	437.81	691.50	1109.77	1540.63	1983.01
0.40	575.55	1002.78	1438.64	1882.52	2333.89

This table presents option values for credit spread options. Option values are computed for a range of spread levels and spread volatilities. The first tableau presents the parameters and the second presents the option prices.

equations are merely a few lines and do not contain more than a single integral, they are easy to write computer code for, and implementation with a mathematical software package is simple.

7. Extensions

In this section we conclude the paper by showing that the pricing model can be applied in settings beyond those described thus far in the article. The first setting is that of a no-arbitrage model, where the short-rate process is allowed to have time-varying components, allowing for an exact match with the current term structure of interest rates, volatilities, etc. Thus even in settings where calibration of the model to currently observed data must be exact, the general pricing formulas derived above can still be used for pricing popular fixed-income securities. The second setting is the case where the term structure model is no longer exponential affine, but is exponential separable, that is, where log bond prices are given by

$$\log P_t(\tau) = A_1(\tau) r_t + \sum_{i=2}^{q} A_i(\tau) f_i(r_t, \mathbf{x}_t) + B(\tau), \tag{31}$$

where $P_t(\tau)$ represents the price of a bond maturing in τ periods, $A_i(\tau)$ and $B(\tau)$ represent functions of time to maturity only, and $f_i(r_t, \mathbf{x}_t)$ represents (possibly) nonlinear functions of any factors determining bond prices. The generalization here from the exponential affine class is to allow for nonlinear functions of factors, but to restrict the nonlinear structure so that the time-varying component may still be separated out from the factors. In this setting, the pricing formulas derived above continue to hold, but with the factors, $\mathbf{x}_t$, in each pricing formula replaced by their respective nonlinear functions, $f_i(r_t, \mathbf{x}_t)$.

7.1 No-arbitrage models

Exact calibration of pricing models to currently observable data is an important requirement for most practitioners. One class of no-arbitrage models that allows for this defines the short-rate process with time-varying components in the drift, volatility, and/or jump terms and uses these time-varying components as free variables to match up to observed data. Examples of this class of models include Black, Derman, and Toy (1991), Burnetas and Ritchken (1996), and Heath, Jarrow, and Morton (1992). We now show via our jump diffusion example how to use the pricing formulas derived above in the context of such models in which the bond price is also exponential affine.

We first extend the one-factor, jump diffusion example we have been using throughout this article to allow for a time-varying central tendency. The interest rate process is defined by

$$dr = \kappa[\theta(t) - r]\, dt + \sigma\, dW + J_u dN_u(\lambda_u) - J_d dN_d(\lambda_d),$$

where the central tendency is now a time-varying function, $\theta(t)$, rather than a constant. In this case the generalized term structure model is given by

$$\exp[A^*(\tau, \mathbf{b}, d)r_t + C^*(\tau, \mathbf{b}, d)],$$

where

$$\begin{aligned}
A^*(\tau, \mathbf{b}, d) &= \left(a - \frac{d}{k}\right)e^{-\kappa\tau} + \frac{d}{k} \\
C^*(\tau, \mathbf{b}, d) &= -\frac{u_1^2\sigma^2}{4\kappa\tau}(e^{-2\kappa\tau} - 1) + \frac{u_1 u_2 \sigma^2}{\kappa}(e^{-\kappa\tau} - 1) \\
&\quad + \left[\frac{u_2^2\sigma^2}{2} - \lambda_u + \lambda_d\right]\tau \frac{\lambda_u}{\kappa + d\eta_u} \log|(1 + \eta_u u_2)e^{\kappa\tau} - \eta_u u_1| \\
&\quad - \frac{\lambda_u}{\kappa + d\eta_u} \log|(1 + \eta_d u_2)e^{\kappa\tau} - \eta_d u_1| \\
&\quad + c - a + \int_0^\tau \kappa\theta(v)A^*(v, \mathbf{b}, d)dv \\
u_1 &= a - u_2 \\
u_2 &= \frac{d}{\kappa}.
\end{aligned}$$

Here the bond price (formed when $a = 0,\ c = 0,\ d = -1$) is a function of the time-varying function $\theta(t)$, which appears in the expression for $C^*(\tau, \mathbf{b}, d)$. Therefore, by choosing the function for $\theta(t)$ appropriately, the current term structure of interest rates, volatilities, etc., can be matched perfectly. Furthermore, because the price of a bond is exponential affine here, all of the pricing formulas for fixed-income derivatives derived in the article apply with this model as well. Consequently, once calibration of this model to currently observed data is accomplished, pricing formulas for popular fixed-income securities can be written by inspection using the formulas derived earlier.

7.2 Exponential separable models

Considerable research is now being focused on nonaffine term structure models. While few such models have been found with closed-form solutions, we want to allow for the use of the formulas derived above for a certain class of models which seem promising: the exponential separable class. Bond prices for this class of models have the form given in Equation (31). Examples of this class for which closed-form solutions exist include Longstaff (1989) and Constantinides (1992).

It is easy to show that the pricing models derived in this article can easily accomodate term structure models of this class with minor changes. Specifically, we first introduce the new variables y_i, $i = 2, \ldots, q$. These variables are defined as

$$y_2 = f_2(r_t, \mathbf{x}_t)$$
$$y_3 = f_3(r_t, \mathbf{x}_t)$$
$$\vdots$$
$$y_q = f_q(r_t, \mathbf{x}_t).$$

Notice now that the term structure model is now exponential affine in the interest rate and these new variables. Therefore, from Duffie and Kan (1996), we know that the stochastic differential equations governing must have linear drifts and instantaneous variances. Thus the interest rate and the new factors, y_i, now have linear drifts and variances in y_i. Thus the transformed term structure model fits into the exponential affine class of models and all of the pricing results derived in this article apply. Consequently the results of this article apply to exponential separable models as well, but with the individual factors, $\mathbf{x}_t$, in the pricing formulas replaced by their nonlinear function counterparts, $f_i(r_t, \mathbf{x}_t)$, from the generalized exponential separable term structure model.

8. Concluding Comments

Duffie and Kan (1996) established the relationship between affine stochastic processes and bond pricing equations in exponential term structure models. We extend the results in their article to the pricing of interest rate derivatives. This article shows that if an exponential affine structure is assumed for the term structure, there is a fundamental link between the components of the bond pricing solution and the prices of many widely traded interest rate derivative securities.

The intuition for our results stems from the fact that derivative prices are derived from a set of differential equations that are similar to those for bond prices up to a modification of constant terms. Our results apply to multifactor processes with multiple diffusions and jump processes. Regardless of the number of shocks, the pricing solutions require at most a single numerical integral, making the model easy to implement. In addition, we show that the results of the article can be easily extended to no-arbitrage models of the type developed in Heath, Jarrow, and Morton (1992), with time-varying components in the short rate or factors, as well as a class of nonlinear term structure models: exponential separable term structure models, such as that in Constantinides (1992).

We provide many examples of options that yield solutions using the methods of the article. While the general approach is the same, the mathematical details for each option vary, resulting in three separate option models, based on the structure of the payoff function.

Appendix A. Solving for the Probability Functions

In this section we solve for various probability functions needed for the different options priced in this article. Each probability function varies in subtle ways from the other and requires different techniques for their solution. The following subsections are categorized by the structure of the payoff function.

1.1 Linear payoff functions

1.1.1 Solution for $\Pi_{0,t}$ with linear payoffs. To solve for the function $\Pi_{0,t} = E_t[e^{-Z_t(T)}(k_0 r_T + k_1 x_{1,T} + \cdots + k_m x_{m,T} + k_{m+1})]$, we first note that

$$\begin{aligned} &E_t\left[e^{-Z_t(T)}(k_0 r_T + k_1 x_{1,T} + \cdots + k_m x_{m,T} + k_{m+1,T})\right] \\ &\quad = \left\{\frac{\partial}{\partial\phi} E_t\left[e^{-Z_t(T)} e^{\phi(k_0 r_T + k_1 x_{1,T} + \cdots + k_m x_{m,T} + k_{m+1,T})}\right]\right\}_{\phi=0}. \end{aligned}$$

To justify the interchange of the expectation and differentiation operators, we need to impose certain restrictions such that the expectation above is well behaved. Define the linear function $f_T(\mathbf{r}, \mathbf{x}, \hat{\tau}) \equiv k_0 r_T + k_1 x_{1,T} + \cdots + k_m x_{m,T} + k_{m+1,T}$. We require that the drift, diffusion, and jump components of the interest and factors be restricted so that $E(|f_T(\mathbf{r}, \mathbf{x}, \hat{\tau})|^k)$ be bounded for any constant k. Furthermore, the derivative taken in the second line above must be well behaved, and so we require the function $E\{\exp[\phi f_T(\mathbf{r}, \mathbf{x}, \hat{\tau})]\}$ to be uniformly bounded at $\phi = 0$.

With these restrictions established, the justification of the interchange of the expectation and differentiation operators now follows. First, notice that we can rewrite the expectation above in the complex domain:

$$\begin{aligned} &\left\{\frac{\partial}{\partial\phi} E_t\left[e^{-Z_t(T)} e^{\phi(k_0 r_T + k_1 x_{1,T} + \cdots + k_m x_{m,T} + k_{m+1,T})}\right]\right\}_{\phi=0} \\ &\quad = \frac{1}{i}\left\{\frac{\partial}{\partial\phi} E_t\left[e^{-Z_t(T)} e^{i\phi(k_0 r_T + k_1 x_{1,T} + \cdots + k_m x_{m,T} + k_{m+1,T})}\right]\right\}_{\phi=0}. \end{aligned}$$

Since the right-hand side of this equation is the derivation of the first moment from the characteristic function, intuitively the assumption of boundedness on the payoff function should ensure that this expectation is bounded as well. We now show this more formally:

$$\begin{aligned} &\frac{\partial}{\partial\phi} E_t\left[e^{-Z_t(T)} e^{i\phi f_T(\mathbf{r},\mathbf{x},\hat{\tau})}\right] - E_t\left[e^{-Z_t(T)} i f_T(\mathbf{r}, \mathbf{x}, \hat{\tau}) e^{i\phi f_T(\mathbf{r},\mathbf{x},\hat{\tau})}\right] \\ &\quad = \frac{E_t\left[e^{-Z_t(T)} e^{i(\phi+\delta) f_T(\mathbf{r},\mathbf{x},\hat{\tau})}\right] - E_t\left[e^{-Z_t(T)} e^{i\phi f_T(\mathbf{r},\mathbf{x},\hat{\tau})}\right]}{\delta} - E_t\left[e^{-Z_t(T)} i f_T(\mathbf{r}, \mathbf{x}, \hat{\tau}) e^{i\phi f_T(\mathbf{r},\mathbf{x},\hat{\tau})}\right] \\ &\quad = \frac{E_t\left[e^{-Z_t(T)} e^{i\phi f_T(\mathbf{r},\mathbf{x},\hat{\tau})}\left(e^{i\delta f_T(\mathbf{r},\mathbf{x},\hat{\tau})} - 1\right)\right]}{\delta} - E_t\left[e^{-Z_t(T)} i f_T(\mathbf{r}, \mathbf{x}, \hat{\tau}) e^{i\phi f_T(\mathbf{r},\mathbf{x},\hat{\tau})}\right] \\ &\quad = E_t\left[e^{-Z_t(T)} e^{i\phi f_T(\mathbf{r},\mathbf{x},\hat{\tau})} \frac{(e^{i\delta f_T(\mathbf{r},\mathbf{x},\hat{\tau})} - 1 - i\delta f_T(\mathbf{r}, \mathbf{x}, \hat{\tau}))}{\delta}\right]. \end{aligned}$$

The integrand in the final equation can be shown to be dominated by $2|f_T(\mathbf{r}, \mathbf{x}, \hat{\tau})|$ and goes to 0 with δ; therefore the expected value goes to 0 by the dominated convergence theorem, and we have the result that

$$\frac{\partial}{\partial\phi} E_t\left[e^{-Z_t(T)} e^{i\phi f_T(\mathbf{r},\mathbf{x},\hat{\tau})}\right] = E_t\left[e^{-Z_t(T)} i f_T(\mathbf{r}, \mathbf{x}, \hat{\tau}) e^{i\phi f_T(\mathbf{r},\mathbf{x},\hat{\tau})}\right].$$

As a result, to solve for $\Pi_{0,t}$ we need to solve for $E_t[e^{-Z_t(T)}e^{\phi(k_0 r_T + k_1 x_{1,T} + \cdots + k_m x_{m,T} + k_{m+1,T})}] \equiv \Gamma_{0,t}$ and simply evaluate the partial derivative of this expression with respect to ϕ at $\phi = 0$. We now apply the Feynman–Kac relation, from which we know that this equation solves the PDDE

$$0 = \mathscr{D}\Gamma_{0,t} - r_t\Gamma_{0,t} \tag{32}$$

with boundary condition

$$\begin{aligned}\Gamma_{0,T} &= \exp\phi f_T(\mathbf{r}, \mathbf{x}, \hat{\tau}) \\ &= \exp[\phi k_0 r_T + \phi k_1 x_{1,T} + \cdots + \phi k_m x_{m,T} + \phi k_{m+1,T}].\end{aligned} \tag{33}$$

In comparing Equations (32) and (33) with Equations (3) and (6), we see that the PDDEs are exactly the same, while the boundary conditions differ by only a set of constant coefficients in front of the interest rate and factors. Therefore, by analogy, we can write the solution to $\Gamma_{0,t}$ as

$$\Gamma_{0,t} = \exp\left[A^*(\theta;\tau,\mathbf{b}_0,\mathbf{d}^*)r_t + \sum_{i=1}^{m} B_i^*(\theta;\tau,\mathbf{b}_0,\mathbf{d}^*)x_{i,t} + C^*(\theta;\tau,\mathbf{b}_0,\mathbf{d}^*)\right], \tag{34}$$

where

$$\mathbf{b}_0 = \begin{bmatrix} \phi k_0 \\ \phi k_1 \\ \vdots \\ \phi k_m \\ \phi k_{m+1} \end{bmatrix}.$$

Thus the solution for $\Pi_{0,t}$ is given by

$$\begin{aligned}\Pi_{0,t} &= \frac{\partial}{\partial\phi}\Gamma_{0,t}\bigg|_{\phi=0} \\ &= \left\{\Gamma_{0,t}\left[\frac{\partial A^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*)}{\partial\phi}r_t + \sum_{i=1}^{m}\frac{\partial B_i^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*)}{\partial\phi}x_{i,t} + \frac{\partial C^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*)}{\partial\phi}\right]\right\}_{\phi=0}.\end{aligned}$$

1.1.2 Solution for $\Pi_{1,t}$ with linear payoffs. For the linear payoff function in Equation (14), the probability $\Pi_{1,t}$ is given by the expression

$$\begin{aligned}\Pi_{1,t} &= E_t\left[\frac{e^{-Z_t(T)}f_T(\mathbf{r},\mathbf{x},\hat{\tau})1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau})\geq K\}}}{E_t[e^{-Z_t}f_T(\mathbf{r},\mathbf{x},\hat{\tau})]}\right] \\ &= \frac{1}{\Pi_{0,t}}E_t[e^{-Z_t(T)}f_T(\mathbf{r},\mathbf{x},\hat{\tau})1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau})\geq K\}}].\end{aligned}$$

Using the Feynman–Kac relation, one can show that the probability satisfies a PDDE very similar to the bond price equation, but with a discontinuous boundary condition. However, the discontinuity of the boundary condition makes this an extremely difficult equation to solve. Instead, we will first solve for the characteristic function, denoted $\widetilde{\Pi}_{1,t}$, associated with this probability and then invert this characteristic function to obtain the probability. The characteristic function is defined as

$$\widetilde{\Pi}_{1,t} = \frac{1}{\Pi_{0,t}}E_t[e^{-Z_t(T)}f_T(\mathbf{r},\mathbf{x},\hat{\tau})e^{i\omega f_T(\mathbf{r},\mathbf{x},\hat{\tau})}], \tag{35}$$

where $i=\sqrt{-1}$ and ω is a real-valued dummy variable. We can rewrite the characteristic function as follows:

$$\begin{aligned}\widetilde{\Pi}_{1,t} &= \frac{1}{\Pi_{0,t}}\frac{1}{i}\frac{\partial}{\partial\omega}E_t\left[e^{-Z_t}e^{i\omega f_T(\mathbf{r},\mathbf{x},\hat{\tau})}\right]\\ &= \frac{1}{\Pi_{0,t}}\frac{1}{i}\frac{\partial}{\partial\omega}\Gamma_{1,t},\end{aligned} \tag{36}$$

where $\Gamma_{1,t}\equiv E_t[e^{-Z_t}e^{i\omega f_T(\mathbf{r},\mathbf{x},\hat{\tau})}]$.

We justify the interchange of the differentiation and expectation operators. We require that the drift, diffusion, and jump components of the interest rate and factors satisfy the restrictions that $E[|f(\mathbf{r},\mathbf{x},\hat{\tau})|^k]$ be bounded for any k and $E\{\exp[\phi f_T(\mathbf{r},\mathbf{x},\hat{\tau})]\}$ be uniformly bounded at $\phi=0$. The proof now follows:

$$\begin{aligned}&\frac{\partial}{\partial\omega}E_t\left[e^{-Z_t(T)}e^{i\omega f_T(\mathbf{r},\mathbf{x},\hat{\tau})}\right]-E_t\left[e^{-Z_t(T)}if_T(\mathbf{r},\mathbf{x},\hat{\tau})e^{i\omega f_T(\mathbf{r},\mathbf{x},\hat{\tau})}\right]\\ &\quad=\frac{E_t\left[e^{-Z_t(T)}e^{i(\omega+\delta)f_T(\mathbf{r},\mathbf{x},\hat{\tau})}\right]-E_t\left[e^{-Z_t(T)}e^{i\omega f_T(\mathbf{r},\mathbf{x},\hat{\tau})}\right]}{\delta}-E_t\left[e^{-Z_t(T)}if_T(\mathbf{r},\mathbf{x},\hat{\tau})e^{i\omega f_T(\mathbf{r},\mathbf{x},\hat{\tau})}\right]\\ &\quad=\frac{E_t\left[e^{-Z_t(T)}e^{i\omega f_T(\mathbf{r},\mathbf{x},\hat{\tau})}\left(e^{i\delta f_T(\mathbf{r},\mathbf{x},\hat{\tau})}-1\right)\right]}{\delta}-E_t\left[e^{-Z_t(T)}if_T(\mathbf{r},\mathbf{x},\hat{\tau})e^{i\omega f_T(\mathbf{r},\mathbf{x},\hat{\tau})}\right]\\ &\quad=E_t\left[e^{-Z_t(T)}e^{i\omega f_T(\mathbf{r},\mathbf{x},\hat{\tau})}\frac{(e^{i\delta f_T(\mathbf{r},\mathbf{x},\hat{\tau})}-1-i\delta f_T(\mathbf{r},\mathbf{x},\hat{\tau})}{\delta}\right].\end{aligned}$$

The integrand in the final equation can be shown to be dominated by $2|f_T(\mathbf{r},\mathbf{x},\hat{\tau})|$ and goes to 0 with δ; therefore the expected value goes to 0 by the dominated convergence theorem, and we have the result that

$$\frac{\partial}{\partial\omega}E_t\left[e^{-Z_t(T)}e^{i\omega f_T(\mathbf{r},\mathbf{x},\hat{\tau})}\right]=E_t\left[e^{-Z_t(T)}if_T(\mathbf{r},\mathbf{x},\hat{\tau})e^{i\omega f_T(\mathbf{r},\mathbf{x},\hat{\tau})}\right].$$

Note that $\Gamma_{1,t}$ is equivalent to $\Gamma_{0,t}$ evaluated at $\phi=i\omega$. Therefore, from the solution for $\Gamma_{0,t}$ in Equation (34), we have the solution for $\Gamma_{1,t}$:

$$\Gamma_{1,t}=\exp\left[A^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*)r_t+\sum_{i=1}^{m}B_i^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*)x_{i,t}+C^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*)\right], \tag{37}$$

where

$$\mathbf{b}_1=\begin{bmatrix} i\omega k_0\\ i\omega k_1\\ \vdots\\ i\omega k_{m+1}\end{bmatrix}.$$

Then we have the solution for $\widetilde{\Pi}_{1,t}$:

$$\widetilde{\Pi}_{1,t}=\frac{1}{\Pi_{0,t}}\left\{\frac{1}{i}\Gamma_{1,t}\left[\frac{\partial A^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*)}{\partial\omega}r_t+\sum_{i=1}^{m}\frac{\partial B_i^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*)}{\partial\omega}x_{i,t}+\frac{\partial C^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*)}{\partial\omega}\right]\right\}. \tag{38}$$

1.1.3 Solution for $\Pi_{2,t}$ with linear payoffs. For the linear payoff function in Equation (14), the probability $\Pi_{2,t}$ is given by the expression

$$\begin{aligned}\Pi_{2,t} &= E_t\left[\frac{e^{-Z_t(T)}1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau})\geq K\}}}{E_t[e^{-Z_t}]}\right]\\ &= \frac{1}{P_t(\tau)}E_t[e^{-Z_t(T)}1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau})\geq K\}}].\end{aligned}$$

It is easy to show that $\Pi_{2,t}$ satisfies a PDDE very similar to the bond price equation, but as with $\Pi_{1,t}$ above, this PDDE has a discontinuity in its boundary condition which makes the equation extremely difficult to solve. Therefore, just as we solved for $\Pi_{1,t}$, we will solve for $\widetilde{\Pi}_{2,t}$ by first calculating the characteristic function $\Pi_{2,t}$ associated with $\Pi_{2,t}$ and then invert this characteristic function to obtain $\Pi_{2,t}$. The characteristic function is defined as

$$\widetilde{\Pi}_{2,t} = \frac{1}{P_t(\tau)}E_t[e^{-Z_t(T)}e^{i\omega f_T(\mathbf{r},\mathbf{x},\hat{\tau})}].$$

However, $E_t[e^{-Z_t(T)}e^{i\omega f_T(\mathbf{r},\mathbf{x},\hat{\tau})}]$ was calculated above in the derivation for $\Pi_{1,t}$:

$$E_t\left[e^{-Z_t(T)}e^{i\omega f_T(\mathbf{r},\mathbf{x},\hat{\tau})}\right] = \Gamma_{1,t}.$$

Thus we have the result

$$\begin{aligned}\widetilde{\Pi}_{2,t} &= \frac{1}{P_t(\tau)}\Gamma_{1,t}\\ &= \frac{1}{P_t(\tau)}\exp\left[A^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*)r_t + \sum_{i=1}^{m}B_i^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*)x_{i,t} + C^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*)\right],\end{aligned}$$

where

$$\mathbf{b}_1 = \begin{bmatrix} i\omega k_0 \\ i\omega k_1 \\ \vdots \\ i\omega k_m \\ i\omega k_{m+1}\end{bmatrix}.$$

2.2 Exponential linear payoffs

2.2.1 Solution for $\Pi_{0,t}$ with exponential linear payoffs. From Equation (13),

$$\begin{aligned}\Pi_{0,t} &= E_t\left[e^{-Z_t(T)}f_T(\mathbf{r},\mathbf{x},\hat{\tau})\right]\\ &= E_t\left[e^{-Z_t(T)}\exp(k_0 r_T + k_1 x_{1,T} + \cdots + k_m x_{m,T} + k_{m+1,T})\right].\end{aligned}$$

We now apply the Feynman–Kac relation, from which we know that this equation solves the PDDE

$$0 = \mathscr{D}\Pi_{0,t} - r_t\Pi_{0,t} \tag{39}$$

with boundary condition

$$\Pi_{0,T} = \exp f_T(\mathbf{r},\mathbf{x},\hat{\tau}) = \exp[k_0 r_T + k_1 x_{1,T} + \cdots + k_m x_{m,T} + k_{m+1,T}]. \tag{40}$$

Notice that this is the exact same PDDE and boundary condition as Equations (32) and (33) with $\phi = 1$. Therefore we can write the solution for $\Pi_{0,t}$ as

$$\Pi_{0,t} = \exp\left[A^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*)r_t + \sum_{i=1}^{m} B_i^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*)x_{i,t} + C^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*)\right], \tag{41}$$

where

$$\mathbf{b}_0 = \begin{bmatrix} k_0 \\ k_1 \\ \vdots \\ k_m \\ k_{m+1} \end{bmatrix}.$$

2.2.2 Solution for $\Pi_{1,t}$ with exponential linear payoffs. For the exponential linear payoff function in Equation (19), the probability $\Pi_{1,t}$ is given by the expression

$$\begin{aligned}
\Pi_{1,t} &= E_t\left[\frac{e^{-Z_t(T)} f_T(\mathbf{r}, \mathbf{x}, \hat{\tau}) 1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau}) \geq K\}}}{E_t\left[e^{-Z_t} f_T(\mathbf{r}, \mathbf{x}, \hat{\tau})\right]}\right] \\
&= \frac{1}{\Pi_{0,t}} E_t\left[e^{-Z_t(T)} f_T(\mathbf{r}, \mathbf{x}, \hat{\tau}) 1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau}) \geq K\}}\right] \\
&= \frac{1}{\Pi_{0,t}} E_t\left[e^{-Z_t(T)} f_T(\mathbf{r}, \mathbf{x}, \hat{\tau}) 1_{\{\log f_T(\mathbf{r},\mathbf{x},\hat{\tau}) \geq \log K\}}\right].
\end{aligned}$$

As with the linear payoff function, $\Pi_{1,t}$ satisfies a PDDE similar to the bond price equation, but with a discontinuous boundary condition. As we did with $\Pi_{1,t}$ in the linear payoff function, we will first solve for the characteristic function, denoted $\widetilde{\Pi}_{1,t}$, associated with this probability and then invert this characteristic function to obtain the probability. The characteristic function is defined as

$$\begin{aligned}
\widetilde{\Pi}_{1,t} &= \frac{1}{\Pi_{0,t}} E_t\left[e^{-Z_t(T)} f_T(\mathbf{r}, \mathbf{x}, \hat{\tau}) e^{i\omega \log f_T(\mathbf{r},\mathbf{x},\hat{\tau})}\right] \\
&= \frac{1}{\Pi_{0,t}} E_t\left[e^{-Z_t(T)} \exp\left[(1+i\omega)(k_0 r_T + k_1 x_{1,T} + \cdots + k_m x_{m,T})\right]\right].
\end{aligned} \tag{42}$$

From the Feynman–Kac relation, it is easy to see that $\Pi_{0,t}\widetilde{\Pi}_{1,t}$ satisfies

$$0 = \mathscr{D}\Pi_{0,t}\widetilde{\Pi}_{1,t} - r_t \Pi_{0,t}\widetilde{\Pi}_{1,t} \tag{43}$$

with boundary condition

$$\Pi_{0,T}\widetilde{\Pi}_{1,T} = \exp[(1+i\omega)(k_0 r_T + k_1 x_{1,T} + \cdots + k_m x_{m,T} + k_{m+1,T})]. \tag{44}$$

This is the same PDDE and boundary condition as Equations (32) and (33) with $\phi = 1 + i\omega$. Therefore we can write the solution for $\widetilde{\Pi}_{1,t}$ as

$$\widetilde{\Pi}_{1,t} = \frac{1}{\Pi_{0,t}} \exp\left[A^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*)r_t + \sum_{i=1}^{m} B_i^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*)x_{i,t} + C^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*)\right], \tag{45}$$

where

$$\mathbf{b}_0 = \begin{bmatrix} (1+i\omega)k_0 \\ (1+i\omega)k_1 \\ \vdots \\ (1+i\omega)k_m \\ (1+i\omega)k_{m+1} \end{bmatrix}.$$

2.2.3 Solution for $\Pi_{2,t}$ with exponential linear payoffs. For the exponential linear payoff function in Equation (14), the probability $\Pi_{2,t}$ is given by the expression

$$\begin{aligned}\Pi_{2,t} &= E_t\left[\frac{e^{-Z_t(T)}1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau})\geq K\}}}{E_t[e^{-Z_t(T)}]}\right] \\ &= \frac{1}{P_t(\tau)}E_t\left[e^{-Z_t(T)}1_{\{\log f_T(\mathbf{r},\mathbf{x},\hat{\tau})\geq \log K\}}\right].\end{aligned}$$

It is easy to show that $\Pi_{2,t}$ satisfies a PDDE very similar to the bond price equation, but as with $\Pi_{1,t}$ above, this PDDE has a discontinuity in its boundary condition, which makes the equation extremely difficult to solve. Therefore, just as we solved for $\Pi_{1,t}$, we will solve for $\widetilde{\Pi}_{2,t}$ by first calculating the characteristic function, $\Pi_{2,t}$, associated with $\Pi_{2,t}$ and then inverting this characteristic function to obtain $\Pi_{2,t}$. The characteristic function is defined as

$$\begin{aligned}\widetilde{\Pi}_{2,t} &= \frac{1}{P_t(\tau)}E_t\left[e^{-Z_t(T)}e^{i\omega\log f_T(\mathbf{r},\mathbf{x},\hat{\tau})}\right] \\ &= \frac{1}{P_t(\tau)}E_t\left[e^{-Z_t(T)}\exp[i\omega(k_0 r_T + k_1 x_{1,T} + \cdots + k_m x_{m,T} + k_{m+1,T})]\right].\end{aligned}$$

From the Feynman–Kac relation, it is easy to see that $P_t(\tau)\widetilde{\Pi}_{1,t}$ satisfies

$$0 = =P_t(\tau)\widetilde{\Pi}_{2,t} - r_t P_t(\tau)\widetilde{\Pi}_{2,t} \tag{46}$$

with boundary condition

$$P_T(0)\widetilde{\Pi}_{2,T} = \exp[i\omega(k_0 r_T + k_1 x_{1,T} + \cdots + k_m x_{m,T} + k_{m+1,T})]. \tag{47}$$

This is the same PDDE and boundary condition as Equations (32) and (33) with $\phi = i\omega$. Therefore we can write the solution for $\widetilde{\Pi}_{2,t}$ as

$$\widetilde{\Pi}_{2,t} = \frac{1}{P_t(\tau)}\exp\left[A^*(\theta;\tau,\mathbf{b}_0,\mathbf{d}^*)r_t + \sum_{i=1}^{m}B_i^*(\theta;\tau,\mathbf{b}_0,\mathbf{d}^*)x_{i,t} + C^*(\theta;\tau,\mathbf{b}_0,\mathbf{d}^*)\right], \tag{48}$$

where

$$\mathbf{b}_0 = \begin{bmatrix} i\omega k_0 \\ i\omega k_1 \\ \vdots \\ i\omega k_m \\ i\omega k_{m+1} \end{bmatrix}.$$

3.3 Integro-linear payoffs

3.3.1 Solution for $\Pi_{0,t}$ with integro-linear payoffs. For the integro-linear payoff in Equation (26), $\Pi_{0,t}$ is defined as

$$\Pi_{0,t} = E_t\left[e^{-Z_t(T)} f_T(\mathbf{r}, \mathbf{x}, \hat{\tau})\right] \tag{49}$$

$$= E_t[e^{-Z_t(T)} X_T], \tag{50}$$

where the new variable $X_T = \int_t^T (k_0 r_v + k_1 x_{1,v} + \cdots + k_m x_{m,v} + k_{m+1})\, dv$ represents an expansion of the state space. Using the Feynman–Kac relation, $\Pi_{0,t}$ satisfies the PDDE

$$0 = =\Pi_{0,t} - r_t \Pi_{0,t} + (k_0 r_t + k_1 x_{1,t} + \cdots + k_m x_{m,t} + k_{m+1}) \frac{\partial \Pi_{0,t}}{\partial X_t}$$

with the boundary condition $\Pi_{0,T} = X_T$. To solve this PDDE we make the following observation:

$$E_t\left[e^{-Z_t(T)} X_T\right] = \frac{\partial}{\partial \phi} E_t\left[e^{-Z_t(T)+\phi X_T}\right]\Big|_{\phi=0}$$

$$= \frac{\partial \Phi_t}{\partial \phi}\Big|_{\phi=0} \tag{51}$$

where $\Phi_t = E_t\left[e^{-Z_t(T)+\phi X_T}\right]$ and ϕ is an arbitrary constant.[24,25] Using the Feynman–Kac formula, Φ_t satisfies the following PDDE:

$$0 = =\Phi_t + [(\phi k_0 - 1) r_t + \phi k_1 x_{1,v} + \cdots + \phi k_m x_{m,v} + \phi k_{m+1}] \Phi_t.$$

The boundary condition for this equation is $\Phi_T = 1$. Since this is the same PDDE and boundary condition as Equations (3) and (6), but with $d = \mathbf{d}_0$, where

$$\mathbf{d}_0' = \begin{bmatrix} \phi k_0 - 1 \\ \phi k_1 \\ \vdots \\ \phi k_m \\ \phi k_{m+1} \end{bmatrix},$$

we can immediately calculate the solution for Φ_t:

$$\Phi_t = \exp\left[A^*(\theta; \tau, \mathbf{0}, \mathbf{d}_0) r_t + \sum_{i=1}^m B_i^*(\theta; \tau, \mathbf{0}, \mathbf{d}_0) x_{i,t} + C^*(\theta; \tau, \mathbf{0}, \mathbf{d}_0)\right]. \tag{52}$$

From Equation (51) we then have the solution for $\Pi_{0,t}$:

$$\Pi_{0,t} = \left\{\Phi_t \times \left[\frac{\partial A^*(\theta; \tau, \mathbf{0}, \mathbf{d}_0)}{\partial \phi} r_t + \sum_{i=1}^m \frac{\partial B_i^*(\theta; \tau, \mathbf{0}, \mathbf{d}_0)}{\partial \phi} x_{i,t} + \frac{\partial C^*(\theta; \tau, \mathbf{0}, \mathbf{d}_0)}{\partial \phi}\right]\right\}_{\phi=0}. \tag{53}$$

[24] Given that $e^{-\phi Y_t - Z_t}$ is bounded in time and the interest rate process $(r, \mathbf{x})$ is strong Markov, the dominated convergence theorem holds. Therefore the application of Fubini's theorem is permitted here.

[25] The justification for interchanging the differentiation and expectation operators follows along the lines of that set out in the solution for $\Pi_{0,t}$ in Section 1.1.1.

3.3.2 Solution for $\Pi_{1,t}$ with integro-linear payoffs. For the integro-linear payoff, $\Pi_{1,t}$ in Equation (27) is defined as

$$\Pi_{1,t} = E_t\left[\frac{e^{-Z_t(T)}Y_t(T)1_{\{Y_t(T)\geq K\}}}{E_t[e^{-Z_t(T)}Y_t(T)]}\right] \tag{54}$$

$$= \frac{1}{\Pi_{0,t}}E_t\left[e^{-Z_t(T)}Y_t(T)1_{\{Y_t(T)\geq K\}}\right]. \tag{55}$$

Using the Feynman–Kac relation, $\Pi_{1,t}$ satisfies a PDDE similar to the bond price equation, but with a discontinuous boundary condition. Therefore we will first solve for the characteristic function, denoted $\widetilde{\Pi}_{1,t}$, associated with this probability and then invert this characteristic function to obtain the probability. The characteristic function is defined as

$$\widetilde{\Pi}_{1,t} = \frac{1}{\Pi_{0,t}}E_t\left[e^{-Z_t(T)}Y_t(T)e^{i\omega Y_t(T)}\right]. \tag{56}$$

Notice, however, that $E_t\left[e^{-Z_t(T)+\phi Y_t(T)}\right]$ was calculated already in the derivation for $\Pi_{0,t}$ for Equation (27). In this derivation $E_t\left[e^{-Z_t(T)+\phi Y_t(T)}\right]$ was defined as Φ_t and solved in Equation (52). Therefore, we can write the solution to $\widetilde{\Pi}_{1,t}$ as[26]

$$\begin{aligned}\widetilde{\Pi}_{1,t} &= \frac{1}{\Pi_{0,t}}\frac{\partial\Phi}{\partial\phi}\bigg|_{\phi=i\omega}\\ &= \frac{1}{\Pi_{0,t}}\left\{\Phi_t\times\left[\frac{\partial A^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)}{\partial\phi}r_t+\sum_{i=1}^{m}\frac{\partial B_i^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)}{\partial\phi}x_{i,t}+\frac{\partial C^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)}{\partial\phi}\right]\right\}_{\phi=i\omega}.\end{aligned} \tag{57}$$

3.3.3 Solution for $\Pi_{2,t}$ with integro-linear payoffs. For the integro-linear payoff function in Equation (26), the probability $\Pi_{2,t}$ is given by the expression

$$\begin{aligned}\Pi_{2,t} &= E_t\left[\frac{e^{-Z_t(T)}1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau})\geq K\}}}{E_t\left[e^{-Z_t(T)}\right]}\right]\\ &= \frac{1}{P_t(\tau)}E_t\left[e^{-Z_t(T)}1_{\{Y_t(T)\geq K\}}\right].\end{aligned}$$

It is easy to show that $\Pi_{2,t}$ satisfies a PDDE very similar to the bond price equation, but as with $\Pi_{1,t}$ above, this PDDE has a discontinuity in its boundary condition, which makes the equation extremely difficult to solve. Therefore, just as we solved for $\Pi_{1,t}$, we will solve for $\widetilde{\Pi}_{2,t}$ by first calculating the characteristic function, $\Pi_{2,t}$, associated with $\Pi_{2,t}$ and then inverting this characteristic function to obtain $\Pi_{2,t}$. The characteristic function is defined as

$$\begin{aligned}\widetilde{\Pi}_{2,t} &= \frac{1}{P_t(\tau)}E_t\left[e^{-Z_t(T)}e^{i\omega Y_t(T)}\right]\\ &= \frac{1}{P_t(\tau)}E_t\left[e^{-Z_t(T)+i\omega Y_t(T)}\right].\end{aligned}$$

[26] The justification for interchanging the differentiation and expectation operators follows along the lines of that set out in the solution for $\widetilde{\Pi}_{1,t}$ in Section 1.1.2.

However, $E_t\left[e^{-Z_t(T)+i\omega Y_t(T)}\right] \equiv \Phi_t$, which has already been calculated in Equation (52). Therefore, we have the solution for the characteristic function:

$$\widetilde{\Pi}_{2,t} = \left.\frac{\Phi_t}{P_t(\tau)}\right|_{\phi=i\omega} \tag{58}$$

$$= \frac{1}{P_t(\tau)} \exp\left[A^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)r_t + \sum_{i=1}^{m} B_i^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)x_{i,t} + C^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)\right]_{\phi=i\omega}. \tag{59}$$

Appendix B. Illustrative Callibration: The Vasicek Model

The interest rate process used is one with constant coefficients,

$$dr_t = \alpha[\beta - r_t]\,dt + \eta\,dW_t,$$

where α is the coefficient of mean reversion, β is the long-run mean of the interest rate, and η is the volatility coefficient for the driving Wiener process dW_t. The estimation problem is a particular version of the system in Equation (29):

$$\min_{\theta} \sum_{t=1}^{T} \sum_{k=1}^{N} \varepsilon_t[\theta(\tau_k)]^2$$

$$\begin{aligned}
\text{subject to } &: \ \theta = \{\alpha, \beta, \eta\} \\
\varepsilon_t[\theta(\tau_k)] &= Y(\tau_k) + \frac{1}{\tau}[A(\tau_k)r_t + C(\tau_k)] \\
\frac{\partial A}{\partial \tau} &= -\alpha A - 1, \quad \forall \tau_k, A(0) = 0 \\
\frac{\partial C}{\partial \tau} &= \frac{1}{2}\eta^2 A^2 + \alpha\beta A, \quad \forall \tau_k, C(0) = 0.
\end{aligned}$$

We employed a panel of monthly data from the well-known McCulloch–Kwon database. This data has zero-coupon yields for several maturities from 1 month to 20 years. We used the period 8/1985–2/1991, because the database for this period is constructed from noncallable bonds and is not confounded with options effects. The estimation exercise took a few seconds and resulted in the following parameter estimates: $[\alpha = 1.6074, \beta = 0.0874, \eta = 0.0408]$. The estimated parameters may be used directly in the pricing of interest rate derivatives.

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