Pricing Credit Derivatives with Rating Transitions

Viral V. Acharya, Sanjiv Ranjan Das, and Rangarajan K. Sundaram

We present a model for pricing risky debt and valuing credit derivatives that is easily calibrated to existing variables. Our approach expands a classical term-structure model to allow for multiple rating classes of debt. The framework has two salient features: (1) it uses a rating-transition matrix as the driver for the default process, and (2) the entire set of rating categories is calibrated jointly, which allows arbitrage-free restrictions across rating classes as a bond migrates among them. We illustrate the approach by applying it to price credit-sensitive notes that have coupon payments linked to the rating of the underlying credit.

The pricing of credit derivatives is approaching modeling maturity. In particular, "reduced form" models—models that attempt to directly describe the arbitrage-free evolution of risky debt values without reference to an underlying firm-value process—have resulted in successful conjoint implementations of term-structure models with default models.[1]

Although all reduced-form models share a common financial-engineering approach, the modeling procedures actually followed contain substantial differences. At least three branches of the literature may be identified.

First, a class of models follows Duffie and Singleton (1999) and Madan and Unal (1998a, 1998b) by taking as the starting point the stochastic process for the occurrence of default and recovery. Implementation of these models is achieved by calibrating or estimating the parameters so that spreads implied by the model match those observed in the data.

Second, a direct approach follows Das and Sundaram (2000) and Schönbucher (1998). In this approach, an arbitrage-free model is described directly for the joint evolution of riskless interest rates and spreads. The only inputs to the model are the current term structure of riskless interest rates and the spreads and the volatilities of these quantities. This direct approach has the advantage of simplicity in input requirements and flexibility in implementation. In particular, the model may be "closed" in a variety of ways by imposing on it any desired specifications for the default process or the recovery-rate process.[2]

Both of these classes are models of risky-debt pricing that are independent of references to bond ratings (i.e., it is as if there were a single rating class).

The third class of reduced-form models explicitly uses rating-transition matrixes as the drivers of the stochastic process for default (see, for example, Arvantis, Gregory, and Laurent 1999, Bielecki and Rutkowski 2000, Das and Tufano 1996, Jarrow, Lando, and Turnbull 1997, Kijima 1998, Kijima and Komoribayashi 1998, and Lando 1998).[3]

We present a discrete-time valuation model that merges the Das–Sundaram approach with a model that incorporates multiple rating classes. As in Das–Sundaram, our model is based on the term-structure model of Heath, Jarrow, and Morton (1990) and extends the HJM model to include risky debt by adding a "forward spread" process to the forward-rate process for default-risk-free bonds. This model requires no restrictions on the correlation between the processes, and the probability of default at any time may depend on the entire history of the process to that point. Our objective is to describe an arbitrage-free lattice for the joint evolution of riskless interest rates and spreads on the risky debt.

Das and Sundaram described the construction of the required pricing lattice when there is only one class of risky debt. This lattice is sufficient for many applications, but it is inappropriate for pricing credit derivatives and other instruments that are based explicitly on an issuer's rating class, such as credit-sensitive notes, in which the coupon

Viral V. Acharya is assistant professor of finance at the London Business School. Sanjiv Ranjan Das is associate professor of finance at Santa Clara University, California. Rangarajan K. Sundaram is associate professor of finance at New York This article was written while Viral Acharya was a doctoral student at New York University.

amount is tied to the issuer's credit rating. In the current paper, we extend the Das–Sundaram framework to modeling the joint evolution of multiple rating classes on the same pricing lattice. The tricky part of this extension comes from a consistency requirement that arises when we attempt to embed all rating categories in a single pricing lattice.

What is this consistency requirement? Because the credit quality or rating of a debt issuer can improve or deteriorate during the life of its debt, the current credit spread on its debt depends not only on the current rating of the issuer but also on all possible rating classes to which the issuer could migrate over the life of the debt. Thus, the current spread summarizes information about future credit spreads on all possible rating classes to which the borrower could migrate. This interdependence of spreads across rating classes implies that calibration of the spread process for a given rating class must be undertaken simultaneously with the calibration of the forward-spread processes for all other rating classes. Formalizing this interdependence and characterizing the joint calibration process (Proposition 2) is a primary contribution of this article.

One aspect of our modeling process bears particular mention. We do not directly model the movement of spreads of each rating class. Rather, we focus on the process for inter-rating spreads (i.e., the spread for that class over the spread for the immediately superior rating class). Of course, the spread over the default-free rate of any given rating class is simply the sum of the inter-rating spreads for that and all higher classes. The advantage of working with inter-rating spreads, rather than directly with spreads over default-free rates, is that, provided only that the inter-rating spreads stay nonnegative, spreads on lower-rated debt will always be higher than those on higher-rated debt. This restriction on the inter-rating spread processes is easier to model than a restriction that credit spreads for a given maturity be monotonically decreasing in credit quality.[4]

Finally, we note that our model requires only easily available information as inputs:

- the default-risk-free yield curve,
- the term structures of credit spreads for each rating class,
- the term structures of volatilities of these quantities, and
- the (statistical) rating-transition probability matrix.

The first three pieces of information are readily available from such providers as Bloomberg, and the last—a standard input in all rating-based models—may be computed from historical data and is available from Moody's Investors Service and Standard & Poor's.

The Model

We developed the model in discrete time because we envisioned a computer implementation for options with American features (i.e., exercisable on any business day during the life of the option) and path dependence. Consider an economy with a finite time interval $(0,T^*)$. Periods are taken to be of length $h > 0$; thus, a typical time point, t, has the form kh for some integer k. First, we assume that at all times t, a full range of default-free zero-coupon bonds trades, as does a full range of risky zero-coupon bonds for each rating category. We also assume that markets are free of arbitrage, so an equivalent martingale measure, Q, exists for this economy;[5] all references to randomness that follow and all expectations are with respect to this measure.

For any given pair of time points (t,T) with $0 \le t \le T \le (T^* - h)$, let $f(t,T)$ denote the forward rate on the default-free bonds applicable to the period $(T,T+h)$; in words, $f(t,T)$ is the rate as viewed from time t for a default-free lending/investment transaction over the interval $(T,T+h)$. All interest rates in the model are expressed in continuously compounded terms. When $t = T$, the rate $f(t,t)$ is called the "short rate" and is denoted by $r(t)$. The forward-rate curve is assumed to evolve according to the process

$$\begin{aligned} f(t+h,T) = {} & f(t,T) + \alpha(t,T)h \\ & + \sigma(t,T)X_0\sqrt{h}, \end{aligned} \tag{1}$$

where

α = drift of the process

σ = volatility

X_0 = a random variable

Both α and σ may depend on other information available at t, such as the time-t forward rates. To keep notation simple, we have suppressed this possible dependence.

We assume there are $K+1$ rating classes, indexed by $k = 1, \ldots, K+1$. Credit quality deteriorates from Class 1 down to Class $K+1$. Class $K+1$ is the "default state"; the assumption is that once a bond is in the default state, it does not trade and its price, net of any recovery upon default, is zero. For $0 \le t \le T \le (T^* - h)$, let $\varphi(t,T) = [\varphi_1(t,T), \ldots, \varphi_k(t,T), \ldots, \varphi_K(t,T)]'$ denote the forward rate on the risky bonds implied from the spot yield curve. The *forward inter-rating spreads* are defined as the spreads, $s_k(t,T)$, between successive rating categories. These compose a vector,

$$\mathbf{s}(t,T) = [s_1(t,T), \ldots, s_k(t,T), \ldots, s_K(t,T)]', \tag{2}$$

where $s_1 = \varphi_1 - f$, $s_2 = \varphi_2 - \varphi_1$, and so on. Of course, the forward spreads on risky debt are related to the inter-rating spreads as

$$\varphi_k(t,T) = f(t,T) + s_1(t,T) + \ldots + s_k(t,T), \ \forall k. \tag{3}$$

As long as $s_k(t,T)$ is greater than zero for all k, credit spreads assuredly increase as quality level decreases.

Next, we make assumptions about the evolution of the forward inter-rating spreads (and thus of the forward rates on the risky bonds). We take these spreads to follow the process

$$\begin{aligned} s_k(t+h,T) &= s_k(t,T) + \beta_k(t,T)h \\ &\quad + \eta_k(t,T)'\mathbf{X}\sqrt{h}, \ \forall k, \end{aligned} \tag{4}$$

where $\beta_k \in \mathbf{R}$ and $\eta_k \in \mathbf{R}^L$ are, respectively, the drift and volatility coefficients of the process and the variables $\mathbf{X} \in \mathbf{R}^L$ are (possibly correlated) random variables. We define

$$\beta(t,T) = [\beta_1(t,T), \ldots, \beta_k(t,T), \ldots, \beta_K(t,T)]' \in \mathbf{R}^K \tag{5a}$$

and

$$\eta(t,T) = [\eta_1(t,T), \ldots, \eta_k(t,T), \ldots, \eta_K(t,T)] \in \mathbf{R}^{L(K)}. \tag{5b}$$

Note that L, the dimension of the space of diffusion factors that affect the spread processes, can generally be smaller than, greater than, or equal to K. Both β and η may depend on other information available at t. (At this point, we place no restrictions on the joint distribution of X_0 and $\mathbf{X}$. When illustrating implementation of the model in a later section, we assume that the random variables X_0 and $\mathbf{X}$ take values in a discrete state-space.)

We denote by $P(t,T)$ the time-t price of a default-free zero-coupon bond of maturity $T \geq t$ and by $\Pi_k(t,T)$ its risky counterpart in the kth rating class. Then, by definition, we have

$$P(t,T) = \exp\left[-\sum_{i=t/h}^{(T/h)-1} f(t,ih)h\right] \tag{6a}$$

and

$$\Pi_k(t,T) = \exp\left[-\sum_{i=t/h}^{(T/h)-1} \varphi_k(t,ih)h\right], \ \forall k. \tag{6b}$$

Default is modeled through the use of a Markov chain that governs the transitions of each bond from one rating level to another in a period of length h. We denote this Markov chain

$$D = \begin{bmatrix} q_{1,1} & \cdots \cdots & q_{1,K+1} \\ : & & : \\ q_{K,1} & & q_{K,K+1} \\ q_{K+1,1} & \cdots \cdots & q_{K+1,K+1} \end{bmatrix}. \tag{7}$$

Because we have taken default to be an absorbing state, we can write

$$D = \begin{bmatrix} q_{1,1} & \cdots \cdots & q_{1,K+1} \\ : & & : \\ q_{K,1} & & q_{K,K+1} \\ 0 \quad 0 & \ldots & 1 \end{bmatrix}. \tag{8}$$

The elements of D depend, of course, on the size of time step h. Moreover, they could be functions of the information set as well as time. To reduce the notational burden, we suppress this dependence. As in Jarrow, Lando, and Turnbull, who first explored this idea of using rating-transition matrixes as a modeling input, we assume that the rating-transition process is independent of the stochastic processes driving the evolution of the model's forward rates.

The spreads on the risky bonds represent the cost of default, and as such, they depend on the probability of default (which depends, in turn, on the sequence of rating transitions until maturity of the bond) and on the amount that bondholders expect to recover in the event of default. Given that default has not occurred up to t, we use $\lambda_k(t) \equiv q_{k,K+1}(t)$ to denote the probability of default by time $t + h$ from state k. As for recovery in the event of default, we use the "recovery of market value" (RMV) convention of Duffie and Singleton, which expresses the recovery rate as a fraction of the market value that would have prevailed in the absence of default. Specifically, Φ^t denotes the actual recovery amount in the event of default at t. The RMV convention then states that, conditional on default occurring at time $t + h$, the time-t expectation, $E^t(\Phi^{t+h})$, of the amount bondholders will receive is

$$E^t(\Phi^{t+h}) = \phi_k(t)E^t[\Pi(t+h,T) \mid \text{No default}], \tag{9}$$

where recovery rate $\phi_k(t)$ depends on the state k from which default occurred. Recovery rates may be chosen to be different, depending on the rating class from which the bond has moved to the default state. As with λ, $\phi_k(t)$ may also depend on all information in the model up to and including period t. It may also depend on the subordination level of the bond. The recovery rate may not be specific to the initial rating class because eventually all defaulting bonds are in the default category.

The following preliminary result relating short spreads to the default probabilities and recovery rates under Q will come in handy in the rest of the article:

$$\sum_{j=1}^{k} s_j(t, t) = -\frac{1}{h}\ln[1 - \lambda_k(t) + \lambda_k(t)\phi_k(t)], \forall k. \quad (10)$$

To understand Equation 10, consider a risky bond at t that matures at $t + h$. By definition, its time-t price is given by

$$\Pi_k(t, t+h) = \exp\left\{-\left[f(t,t) + \sum_{j=1}^{k} s_j(t,t)\right]h\right\}, \forall k. \quad (11)$$

Now, a one-period investment in this bond fetches a cash flow of \$1 at time $t + h$ and a cash flow of $\phi_k(t)$ if it defaults. When discounted at the short rate, the expected cash flow (in the risk-neutral world) must equal the initial price of the bond, so we obtain

$$\Pi_k(t, t+h) = \exp[-f(t,t)h][1 - \lambda_k(t) + \lambda_k(t)\phi_k(t)], \forall k. \quad (12)$$

Equation 10 is an immediate consequence of Equations 11 and 12.

The model's objective is to develop a risk-neutral lattice for pricing risky debt. We pursue this purpose in several steps. First, we generate the lattice of default-free interest rates by solving for the risk-neutral drifts so that all discounted default-free securities are martingales. Then, we superimpose a lattice for credit spreads on the first lattice and compute the risk-neutral drifts for the forward-spread process so that the discounted prices of risky debt are martingales. Finally, we use the recursive structure of the model, together with a specific assumption concerning the default process, to illustrate implementation of the model. We begin with identification of the risk-neutral drifts.

Identifying the Risk-Neutral Drifts

In this section, we derive recursive expressions for drifts α and β of, respectively, the forward-rate and spread processes in terms of volatilities σ and η. To this end, we define $B(t)$ as the time-t value of a "money market account" that uses an initial investment of \$1 and rolls the proceeds over at the default-free short rate:

$$B(t) = \exp\left[\sum_{i=0}^{(t/h)-1} r(ih)h\right]. \quad (13)$$

We assume, without loss of generality, that equivalent martingale measure Q was defined with respect to $B(t)$ as *numeraire*; thus, under Q, all asset prices in the economy discounted by $B(t)$ will be martingales.

We first identify the risk-neutral drifts, α, of the default-free forward rates under Q. A well-known property of the HJM framework is that these risk-neutral drifts can be expressed entirely in terms of the forward-rate volatilities, σ. Thus, to be precise, we have the first proposition:

Proposition 1 (drift of the default-free forward-rate process). For any t, the following recursive relationship holds between drifts α and volatilities σ:

$$\sum_{i=(t/h)+1}^{(T/h)-1} \alpha(t, ih) = \left(\frac{1}{h^2}\right)\ln\left(E^t\left\{\exp\left[-\sum_{i=(t/h)+1}^{(T/h)-1} \sigma(t, ih)X_0 h^{3/2}\right]\right\}\right) \quad (14)$$

The proof is in Appendix A.

The next step is to obtain an analog of Proposition 1 for drifts $\beta(t,T)$ of the forward inter-rating spread processes in terms of their volatilities. This representation is a bit trickier, however, than the representation for the default-free rates. A risky bond with a current rating of k may move to a different rating class tomorrow. Thus, the current price of the bond (equivalently, the spread for its rating class) implicitly also carries information about future spreads associated with other rating classes. This implies, in turn, the presence of simultaneous no-arbitrage restrictions on how the drifts of various classes evolve with respect to each other. The following result unravels this dependence and shows how the relevant drifts may be calculated in a bootstrapping manner.

Proposition 2 (drifts of the forward inter-rating spread processes). For $j = 1, \ldots, K$, let θj be defined as

$$\theta_j(t, ih) = \sum_{l=1}^{j} \beta_l(t, ih). \quad (15)$$

Then, at each t, the vector $[\theta_1(t,ih), \ldots, \theta_K(t,ih)]$ must solve the following system of K unknowns (x_j, $j = 1, \ldots, K$):

$$\sum_{j=1}^{K} a_{k,j} b_{k,j} x_j = 1, \text{ for } k = 1, \ldots, K, \quad (16)$$

where

$$a_{k,j} = \frac{q_{k,j}(t)}{1-\lambda_k(t)} \times \exp\left[-\sum_{i=(t/h)+1}^{(T/h)-1} \alpha(t, ih)h^2\right] \times \exp\left\{-\sum_{i=(t/h)+1}^{(T/h)-1} [\varphi_j(t, ih) - \varphi_k(t, ih)]h\right\}, \tag{17a}$$

$$b_{k,j} = E^t\left(\exp\left\{-\sum_{i=(t/h)+1}^{(T/h)-1}\left[\sigma(t, ih)X_0 + \sum_{l=1}^{j} \eta_l(t, ih)'\mathbf{X}\right]h^{3/2}\right\}\right), \tag{17b}$$

and

$$x_j = \exp\left[-\sum_{i=(t/h)+1}^{(T/h)-1} \theta_j(t, ih)h^2\right]. \tag{17c}$$

For the proof, see Appendix A.

These expressions are much less forbidding than they may first appear. The system is linear in the x_j variables. At each state at time t in rating class k, one can compute the $a_{k,j}$ terms if one knows the transition probabilities, the α drift terms, and the spread levels at that state. Similarly, the $b_{k,j}$ terms can be computed by taking the expectation over diffusion processes (X_0, $\mathbf{X}$) if one knows the volatilities of the term structure of the forward interest rate and of the forward inter-rating spread. Thus, when we solve this system of linear equations (Equation 16), we obtain x_j terms, which in turn, yield θ_j terms.[6] Because the system is solved in a bootstrap manner starting with $T-1$, the drift terms, $\beta_j(t,\bullet)$ can then be backed out from the knowledge of $\theta_j(\bullet,\bullet)$.

The presence of multiple rating classes prevents this representation from providing an analytical expression for drift terms $\beta(t,T)$, as is obtained in the single-rating model of Das and Sundaram. However, the expectation in Equation 17b over all possible sample paths of the state space for X_0 and $\mathbf{X}$ can be computed numerically by using a lattice, as we illustrate in this article. The result is the derivation of the risk-neutral drifts in terms of the volatilities.

Recursive Representation of Risky-Bond Prices

In our model, as in Das and Sundaram, risky-bond prices have a recursive representation that leads, in turn, to a representation in terms of bond prices of short maturities [i.e., of the form $\Pi_k(\tau,\tau+h)$]. We describe this representation in this section. Whereas in Das and Sundaram, the recursive representation entails one level of recursion at each time step, with possible rating transitions, our representation forks into K levels of recursion at each time step.

It is straightforward to show (see Equation A12 in Appendix A) that

$$\exp[-\varphi_k(t, t)h] \times E_k^t[\Pi(t+h, T)|\text{No default}] = \Pi_k(t, T). \tag{18}$$

Rearranging terms and using the fact that $\exp[-\varphi_k(t, t)h] = \Pi_k(t,t+h)$, we now obtain

$$\begin{aligned}\Pi_k(t, T) &= \Pi_k(t, t+h)E_k^t[\Pi(t+h, T)|\text{No default}] \\ &= \Pi_k(t, t+h)\sum_{j=1}^{K}\frac{q_{k,j}(t)}{1-\lambda_k(t)}E^t[\Pi_j(t+h, T)].\end{aligned} \tag{19}$$

We can now iterate on the expression for $\Pi_j(t+h,T)$ in terms of the transition probabilities $q_{j,l}(t+h)$ and $E^{t+h}[\Pi_l(t+h,T) \mid \text{No default}]$, with $l = 1, \ldots, K$.

The recursive structure of the prices of risky bonds, as described in Equation 19, facilitates computation of these prices. Note that, because all terms on the right-hand side have the form $F(\tau,\tau+h)$, we can use the relationship in Equation 12 to employ the forward-spread components (i.e., the default and recovery rates) in this process.

Implementation of the Model

To implement the model, we must be more precise about quantities that have so far been left unspecified, namely, the random variables X_0 and $\mathbf{X}$. In this section, we describe the assumptions that we use in the rest of this article. We chose the assumptions with an eye toward simplicity, both in exposition and in implementation, but they are primarily meant to be illustrative; alternative assumptions could be similarly handled.

We first assume that $K=2$, so the three possible states of the corporate bond are investment grade ($k=1$), speculative grade ($k=2$), and default state ($k=3$). We make the discrete time assumption that X_0 and $\mathbf{X}$ (i.e., X_0, X_1, and X_2) are binomial random variables—specifically, that each takes on the value ± 1 with probability 1/2. We assume that the pairwise correlation between X_0 and X_1 is ρ_1, between X_0 and X_2 is ρ_2, and between X_1 and X_2 is ρ_3. So, the assumed joint distribution of X_0, X_1, X_2 is[7]

$$(X_0, X_1, X_2) = \begin{cases} (+1,+1,+1) \text{ w.p. } q_{uuu} = (1+\rho_1+\rho_2+\rho_3)/8 \\ (+1,+1,-1) \text{ w.p. } q_{uud} = (1+\rho_1-\rho_2-\rho_3)/8 \\ (+1,-1,+1) \text{ w.p. } q_{udu} = (1-\rho_1+\rho_2-\rho_3)/8 \\ (+1,-1,-1) \text{ w.p. } q_{udd} = (1-\rho_1-\rho_2+\rho_3)/8 \\ (-1,+1,+1) \text{ w.p. } q_{duu} = (1-\rho_1-\rho_2+\rho_3)/8 \\ (-1,+1,-1) \text{ w.p. } q_{dud} = (1-\rho_1+\rho_2-\rho_3)/8 \\ (-1,-1,+1) \text{ w.p. } q_{ddu} = (1+\rho_1-\rho_2-\rho_3)/8 \\ (-1,-1,-1) \text{ w.p. } q_{ddd} = (1+\rho_1+\rho_2+\rho_3)/8 \end{cases} \quad (20)$$

In general, the correlation coefficients may not equal zero or even be constant over the tree.[8]

Next, we look at the components of the forward rates, namely, default rate $\lambda_k(t)$ and recovery rate $\phi_k(t)$. Using Equation 10 shows clearly that knowing the forward spreads, $s_k(t,t)$, and either $\lambda_k(t)$ or $\phi_k(t,t)$ for all k will allow us to infer the other. Unlike Das and Sundaram, where an additional specification is required to link default rate $\lambda(t)$ to the interest rate and the spread variables, in our model, the $\lambda_k(t)$'s to be used are based on the rating-transition matrix. In particular, $\lambda_k(t) \equiv q_{k,K+1}(t)$.

One last and nontrivial issue remains before we can discuss the engineering details of model implementation. Estimates of the probabilities provided in standard rating-transition matrixes based on historical data (such as the estimates of Moody's and Standard and Poor's) cannot be used directly in our model because our model [including the probability of default, $\lambda_k(t)$] is set in the risk-neutral world. Thus, a translation from the actual to the risk-neutral measure is required. To this end, we suppose that $\lambda_k^P(t)$ denotes the actual probability of default. Then, we make the natural assumption that the recovery rates are the same in the risk-neutral and actual worlds. So, realized cash flows coincide in the two cases. Letting $\xi_k(t)$ be the time-t premium for bearing default risk corresponding to rating state k, the analog of Equation 10 under the actual probabilities is easily derived to be

$$\exp[-s_k(t,t)h] = \exp[-\xi_k(t)h] \times [1 - \lambda_k^P(t) + \phi_k(t)\lambda_k^P(t)]. \quad (21)$$

The difference between Equations 10 and 21 is simply that the relationship described in Equation 10 was developed in the risk-neutral world, where (by definition) there is no premium for bearing risk. Expression 21 follows the same derivation but is set in the actual world, where we would expect the risk-premium term, $\xi_k(t)$, to be positive.

Comparing Equations 10 and 21, we note that we may express $\lambda_k(t)$ in terms of $\lambda_k^P(t)$ and risk premium $\xi_k(t)$ as follows:

$$\lambda_k(t) = \lambda_k^P(t)\left(\frac{1-\exp[-s_k(t,t)h]}{1-\exp\{-[s_k(t,t)-\xi_k(t)]h\}}\right). \quad (22)$$

Expression 22 implies the intuitive condition that λ_k is greater than λ_k^P whenever risk-premium ξ_k is positive.

Equations 21 and 22 may be used in conjunction with Equation 10 to estimate risk-premium term $\xi_k(t)$. Specifically, the result is

$$\phi_k(t) = \left[\frac{1}{\lambda_k^P(t)}\right]\exp\{-[s_k(t,t)-\xi_k(t)]h\} - 1 + \lambda_k^P(t). \quad (23)$$

In estimation, we can use $\phi_k(t)$ to be the average recovery rate observed historically for the rating class k, $\bar{\phi}_k(t)$. Thus, knowing actual recovery rate $\bar{\phi}_k(t)$, actual default rate $\lambda_k^P(t)$, and actual spot spread $s_k(t,t)$, we can use Equation 23 to back out risk-premium term $\xi_k(t)$. Or, as in Das and Sundaram, we can assume that the risk-premium term is given by $\xi_k(t,t) = \nu_k s_k(t,t)$ for scalar ν_k and use Equation 23 to back out implied recovery rate function $\phi_k(t)$.

An additional complication remains—that is, to adjust the remaining elements of the historical transition matrix to obtain the risk-neutral transition matrix. For this task, we make an assumption, similar to that in Jarrow, Lando, and Turnbull, that

$$q_{k,l}(t) = \delta_k(t)q_{k,l}^P(t), \forall l \neq k \quad (24a)$$

and

$$q_{k,k}(t) = 1 + \delta_k(t)[q_{k,k}^P(t) - 1], \quad (24b)$$

where

$$\delta_k(t) = \frac{\lambda_k(t)}{\lambda_k^P(t)}. \quad (24c)$$

Note that $q_{k,l}(t)$ refers to transition probabilities in the risk-neutral matrix whereas $q_{k,l}^P(t)$ refers to transition probabilities in the historical matrix. Also, $q_{k,K+1}^P(t) = \lambda_k^P(t)$, so $q_{k,K+1}(t) = \lambda_k(t)$. Wilson (1997) also used such a "spread" transformation, where the mass spread from the diagonal terms of the transition matrix to the off-diagonal terms. More sophisticated techniques for estimating $\delta_k(t)$ would try to minimize error over all the transition matrix data rather than only the default transition probability, $\lambda_k(t)$, as we did in the possible estimation techniques described in the previous paragraph.

Lattice Implementation

We describe in this section the use of a lattice to implement our model. The lattice has a multidimensional structure because it combines the evolution of interest rates and inter-rating spreads for different rating classes. At the same time, the rating-transition process is superimposed on this lattice. This superimposition is straightforward because we have assumed that the rating-transition process is independent of the diffusion processes.

First, we will look only at the multidimensional structure for the interest rate and the spread processes. We assume, as before, three possible rating classes: investment grade (denoted I), speculative or junk grade (denoted J), and the default state (denoted D). Therefore, $K = 2$ and we have two inter-rating spread processes, s_I and s_J. Together with the interest rate process, we thus obtain a triple-binomial structure with eight branches emanating from each node of the lattice. This part of the lattice looks similar to the lattice in Das and Sundaram. Specifically, once the risk-neutral drifts, $\alpha(\bullet)$, $\beta(\bullet)$, have been computed at any t, the possible values of the forward rates and forward spreads one period ahead are readily obtained by using Equations 1 and 4. At each state, the current rating class is also known. Thus, if we are given the forward and spread curves, $F(\tau) = [f(\tau,\bullet)]$, $S_I(\tau) = [s_I(\tau,\bullet]$, and $S_J(\tau) = [s_J(\tau,\bullet)]$ at any τ, and because we know one-period default probability $\lambda(\tau)$ to be the default probability in one period for the current rating class, we can compute recovery rate $\phi(\tau)$ as described in the previous section.

So far, at each node on the lattice, we have information related to all three risks involved in the valuation of risky debt (interest rates, default probabilities, and recovery rates). To obtain the possible one-period-ahead values of risky debt, we need to superimpose the rating-transition process on the lattice as follows: From each of the eight nodes of the triple-binomial spread lattice, three rating transitions emanate. Thus, if the current rating class at the source node is k, three transitions, $k \to I$, $k \to J$, and $k \to D$, are possible and they have probabilities $q_{k,I}$, $q_{k,J}$, and $q_{k,D}$, respectively. From each of the 16 nondefault states so obtained (note that the default state D is an absorbing state), another triple-binomial lattice superimposed on a rating-transition matrix evolves.

Thus, each node carries the information set (F, S_I, S_J, λ_k, ϕ_k), where k is the current rating. As in Das and Sundaram, each node also carries the state price of the node and cumulative default probability up to that node. Thus, we have all the information necessary to price a wide range of standard credit instruments and derivatives.

Figure 1 illustrates the rating-migration process superimposed on the triple-binomial lattice at one of the nodes. The up and down states for the interest rate process, F_u and F_d, correspond to, respectively, $X_0 = +1$ and $X_0 = -1$, with similar notation also used for S_I and S_J. The left part of the branching shows the eight notes that emanate from a starting rating class k, depending on the realizations of the binomial variates, X_0, X_1, X_2. The right part of the branching shows for the up-down-down

Figure 1. Information Generated at Each Node in the Combination Lattice

Note: The probabilities $q_{uuu}, \ldots, q_{ddd}$ for the lattice are given by Equation 20; the probabilities for the rating transitions, $q_{k,I}$, $q_{k,J}$, and $q_{k,D}$, are given by the rating-transition matrix at the corresponding node.

node the rating transitions to the three possible states I, J, and D.

For simplicity, we assume in the calibration that $\sigma(t,T)$, $\eta_I(t,T)$, and $\eta_J(t,T)$ depend only on T.[9] Also, as assumed throughout the text, the correlations between X_0, X_1, and X_2 and rating-transition matrix D are assumed constant. For the numerical example, we consider the calibration exercise for a tree of three periods with the following parameter specifications:

- X_1 and X_2 are assumed to be perfectly correlated ($\rho_3 = 1$).
- The correlations between X_0 and X_1 and between X_0 and X_2 are assumed to be identical, $\rho_1 = \rho_2 = 0.25$.
- The time-step in the tree is $h = 0.5$ (half a year).
- The initial values for the forward risk-free rate and inter-rating spreads and the volatility terms of the forward risk-free rate and inter-rating spread processes are as described in **Table 1**.
- The rating-transition process under the risk-neutral measure is

$$D = \begin{pmatrix} 0.70 & 0.20 & 0.10 \\ 0.10 & 0.75 & 0.15 \\ 0 & 0 & 1 \end{pmatrix}.$$

In this parameter specification, the tree for the evolution of the risk-free forward rate and the inter-rating forward spreads is shown in **Figure 2**. Note that, unlike Figure 1, the rating transitions are not shown as superimposed in this tree, even though the probabilities of these transitions are required for accurate no-arbitrage calibration of the risk-neutral drifts (Proposition 2). In addition, the number of branches has been reduced because the two inter-rating spread processes are assumed to be perfectly correlated. Thus, at the first period, four nodes are possible—*uu*, *ud*, *du*, and *dd*—with probabilities $1/4(1 + \rho)$, $1/4(1 - \rho)$, $1/4(1 - \rho)$, and $1/4(1 + \rho)$—that is, respectively, 0.3125, 0.1875, 0.1875, and 0.3125. From each of these nodes, another four nodes emanate.

At each node in the tree at time ih (the initial node being $i = 0$), the three columns indicate, respectively, the forward risk-free rate, the forward inter-rating spread between the risk-free and investment-grade ratings, and the forward inter-rating spread between the investment-grade and speculative-grade ratings for maturities $(i + 1)h, \ldots, T = 1.5$ years. With these forward rates and using Equations 6a and 6b, we can readily construct the tree for zero-coupon bond prices for a risk-free bond, an investment-grade bond, and a speculative-grade bond. This tree is shown in **Figure 3**. The zero-coupon bond prices constitute the fundamental prices from which other instruments can be priced.

We next illustrate our framework by pricing a credit-related instrument.

Example: Credit-Sensitive Note

We use as our example a credit-sensitive note whose valuation requires modeling both default risk and rating migrations. Other instruments can be priced analogously.

The credit-sensitive note (CSN) is a corporate coupon bond whose coupon is linked to the rating of the corporation. For example, in June 1989, Enron Corporation issued $100 million in noncallable 9.5 percent credit-sensitive notes to mature on June 15, 2001. The coupon on these notes was linked to Enron's credit rating as measured by either Standard & Poor's or Moody's. (At the time of issuance, its outstanding senior debt had ratings of BBB– from Standard & Poor's and Baa3 from Moody's). The coupon on the notes was structured in such a way that if Enron's credit rating changed, the coupon rate would change also. Specifically, the coupon rate was set to drop incrementally for improvements in Enron's ratings and to climb steeply if the rating deteriorated. The exact schedule is in **Table 2**.[10]

We assume that the coupon amount on a coupon payment date is linked to the corporate rating prevailing at the previous coupon payment date. In our model of three rating classes, the CSN has a coupon of c_I for investment-grade rating and c_J for speculative-grade rating. Such a note cannot be priced using a pure spread-based model of credit or a pure intensity-based model of credit. The model described in this paper, however, lends itself appropriately to the valuation of such a CSN.

Table 1. Initial Values

i	$f(t,t+ih)$	$s_I(t,t+ih)$	$s_J(t,t+ih)$	$\sigma(t,t+ih)$	$\eta_I(t,t+ih)$	$\eta_J(t,t+ih)$
1	0.06	0.02	0.04	0.010	0.005	0.005
2	0.07	0.02	0.04	0.011	0.006	0.006
3	0.08	0.03	0.05	0.012	0.006	0.007

Figure 2. Example Tree for Forward Risk-Free Rate and Inter-Rating Spreads

$t = 0$			$t = 0.5$				$t = 1.5$			
							uu	0.092017	0.039013	0.056522
			uu	0.075504	0.023003	0.043004	*ud*	0.092017	0.027013	0.050522
				0.086013	0.033007	0.053515	*du*	0.080017	0.039013	0.056522
							dd	0.080017	0.027013	0.050522
							uu	0.092017	0.033013	0.049522
			ud	0.075504	0.017002	0.037004	*ud*	0.092017	0.021013	0.043522
F	S_I	S_J		0.086013	0.027007	0.046515	*du*	0.080017	0.033013	0.049522
0.06	0.02	0.04					*dd*	0.080017	0.021013	0.043522
0.07	0.02	0.04								
0.08	0.03	0.05								
							uu	0.080017	0.039013	0.056522
			du	0.064504	0.023002	0.043004	*ud*	0.080017	0.027013	0.050522
				0.074013	0.033007	0.053515	*du*	0.068017	0.039013	0.056522
							dd	0.068017	0.027013	0.050522
							uu	0.080017	0.033013	0.049522
			dd	0.064504	0.017002	0.037004	*ud*	0.080017	0.021013	0.043522
				0.074013	0.027007	0.046515	*du*	0.068017	0.033013	0.049522
							dd	0.068017	0.021013	0.043522

Notes: The three columns at each time point indicate, respectively, the forward risk-free rate, the forward inter-rating spread between the risk-free and investment-grade ratings, and the forward inter-rating spread between the investment-grade and speculative-grade ratings for maturities $(i+1)h, \ldots, T = 1.5$ years. The results in Proposition 1 and Proposition 2 were used to calibrate the risk-neutral drifts.

Figure 3. Example Tree for Risk-Free, Investment-Grade, and Speculative-Grade Zero-Coupon Bond Prices

$t = 0$			$t = 0.5$				$t = 1.5$			
							uu	0.955034	0.936585	0.910487
			uu	0.962952	0.951940	0.931690	*ud*	0.955034	0.942221	0.918718
				0.922416	0.896943	0.854684	*du*	0.960781	0.942221	0.915966
							dd	0.960781	0.947892	0.924247
							uu	0.955034	0.939399	0.916424
			ud	0.962952	0.954800	0.937297	*ud*	0.955034	0.945052	0.924709
P	Π_I	Π_J		0.922416	0.902341	0.865435	*du*	0.960781	0.945052	0.921939
0.970446	0.960789	0.941765					*dd*	0.960781	0.950740	0.930274
0.937067	0.918512	0.882497								
0.900325	0.869358	0.818731								
							uu	0.960781	0.942221	0.915966
			du	0.968263	0.957190	0.936829	*ud*	0.960781	0.947892	0.924247
				0.933085	0.907317	0.864570	*du*	0.966563	0.947892	0.921478
							dd	0.966563	0.953596	0.929809
							uu	0.960781	0.945052	0.921939
			dd	0.968263	0.960066	0.942466	*ud*	0.960781	0.950740	0.930274
				0.933085	0.912778	0.875445	*du*	0.966563	0.950740	0.927487
							dd	0.966563	0.956461	0.935873

Note: The underlying forward rate tree is as in Figure 2. The three columns at each time point represent, respectively, prices for a risk-free bond, for an investment-grade bond, and for a speculative-grade bond.

Table 2. Schedule of Coupon Changes

Moody's Rating	S&P Rating	Coupon Rate
Aaa	AAA	9.20%
Aa1 to Aa3	AA+ to AA–	9.30
A1 to A3	A+ to A–	9.40
Baa1 to Baa3	BBB+ to BBB–	9.50
Ba1	BB+	12.00
Ba2	BB	12.50
Ba3	BB–	13.00
B1 or lower	B+ or lower	14.00

The lattice is straightforward. At each node of the lattice, the current rating class is available in the information set at the node. This information determines the coupon payment scheduled for the next coupon payment date. The "upgrading" and "downgrading" along the lattice produce the resetting of the coupon during the life of the schedule as established in the rating schedule. Thus, discounting the cash flows in default and nondefault states and moving backward along the tree yields the appropriate price of the CSN.

To be more precise, by using the recursive implementation discussed in "Recursive Representation of Risky-Bond Prices," we find the price of the credit-sensitive note, $CSN_k(t, T)$, by solving

$$CSN_k(t, T) = \Pi_k(t, t+h) \times \left\{ c_k + \sum_{j=1}^{K} \frac{q_{k,j}(t)}{1-\lambda_k(t)} E^t[CSN_j(t+h, T)] \right\}. \tag{25}$$

Note that $\Pi_k(t, t+h)$ is already available from the zero-coupon bond price tree, c_k represents the coupon for the next period, which is "set" today on the basis of current rating k, and the second term inside {•} in Equation 25 represents the value of the credit-sensitive note at the nodes tomorrow after possible rating migrations (See Note 9).

As an illustration, we consider a variant of the Enron CSN that has 1.5 years to maturity (a three-period note with h = 0.5 year). The coupons for different ratings are $c_I = 0.04675$ and $c_J = 0.06375$, which correspond to semi-annual coupons of 9.35 percent and 12.75 percent, respectively. Using the recursive scheme described previously (or simply reducing the scheme to a backward induction procedure), we find that the CSN can be priced off the zero-coupon bond price tree. The tree for CSN prices is described in **Figure 4**. At each node, the two columns represent the CSN price for, left to right, investment-grade and speculative-grade ratings. For example, at $t = 0$, the CSN price is 0.994146 if the underlying credit has an investment-grade rating, but the price falls to 0.984822 if the underlying credit rating has slipped to speculative.

At t = 1.0 year, the price of the CSN is easy to determine because its coupon payment is "set" for maturity at T = 1.5 years. The price is thus simply equal to

$$(1.0 + c_k)\Pi_k(t, t+h),$$

where k is the current rating of the underlying credit.

Consider now the state of the world uu at t = 0.5 year when the underlying credit has a speculative-grade rating. Its price can be computed by using Equation 25 to be

$$\begin{aligned} CSN_J(t=0.5, T=1.5) &= 0.93169 \\ &\times \left\{ 0.06375 + \frac{0.10}{0.85}\begin{bmatrix} (0.3125)(0.980370) \\ +(0.1875)(0.986270) \\ +(0.1875)(0.986270) \\ +(0.3125)(0.992206) \end{bmatrix} + \frac{0.70}{0.85}\begin{bmatrix} (0.3125)(0.968530) \\ +(0.1875)(0.977286) \\ +(0.1875)(0.974359) \\ +(0.3125)(0.983168) \end{bmatrix} \right\} \\ &= 0.969719. \end{aligned}$$

To show how the credit sensitivity of the coupon payment plays a role in the pricing of the CSN, **Figure 5** shows the tree for prices for a *credit-insensitive* note (CIN) that has a fixed coupon of 0.04675 irrespective of the rating of the underlying credit. The price of this note at t = 0 with investment-grade rating is 0.985483, whereas that of the otherwise identical CSN is 0.994146. The difference in value comes from two sources: (1) At all nodes at t = 0.5 year, if the rating "falls" to speculative grade, the CSN experiences an upward jump in coupon payment from 0.04675 to 0.06375. (2) At all nodes at t = 0.5 year, even if the rating "stays" as investment grade, the price of the CSN will be higher because of the increase in future coupon payments whenever a downgrade occurs. These effects can be observed by comparing the t = 0.5 year prices down the trees in Figure 4 and Figure 5.

Instruments other than credit-sensitive notes that have embedded optionality tied to the credit quality of the underlying security can be priced analogously in a relatively simple manner by using our approach.

Concluding Comments

We developed a model for the pricing of credit derivatives by using observable data. The model is arbitrage free, accommodates path dependence, allows for all rating classes in one consistent lattice framework, and can handle a range of securities that have a credit-related component. Although the model is rich and flexible enough to price any credit-related instrument, it is particularly appropriate for pricing credit-sensitive notes that have payments linked to rating transitions.

Figure 4. Tree for Credit-Sensitive Note's Prices

t = 0		*t* = 0.5			*t* = 1.5		
					uu	0.980370	0.968530
		uu	0.981175	0.969719	ud	0.986270	0.977286
					du	0.986270	0.974359
					dd	0.992206	0.983168
					uu	0.983316	0.974846
		ud	0.987674	0.981145	ud	0.989234	0.983659
CSN_I	CSN_J				du	0.989234	0.980713
0.994146	0.984822				dd	0.995187	0.989579
					uu	0.986270	0.974359
		du	0.992254	0.980576	ud	0.992206	0.983168
					du	0.992206	0.980223
					dd	0.998177	0.989084
					uu	0.989234	0.980713
		dd	0.998828	0.992132	ud	0.995187	0.989579
					du	0.995187	0.986615
					dd	1.001176	0.995534

Note: The two columns at each time point represent, respectively, the CSN price for investment-grade and speculative-grade ratings.

Figure 5. Tree for Credit-*Insensitive* Note's Prices

t = 0		*t* = 0.5			*t* = 1.5		
					uu	0.980370	0.953052
		uu	0.977876	0.941060	ud	0.986270	0.961668
					du	0.986270	0.958787
					dd	0.992206	0.967455
					uu	0.983316	0.959267
		ud	0.984343	0.952229	ud	0.989234	0.967939
CIN_I	CIN_J				du	0.989234	0.965040
0.985483	0.960433				dd	0.995187	0.973764
					uu	0.986270	0.958787
		du	0.988917	0.951681	ud	0.992206	0.967455
					du	0.992206	0.964557
					dd	0.998177	0.973278
					uu	0.989234	0.965040
		dd	0.995459	0.962978	ud	0.995187	0.973764
					du	0.995187	0.970848
					dd	1.001176	0.979625

Note: The two columns at each time point represent, respectively, the price for investment-grade and speculative-grade ratings.

Appendix A. Proofs

Proof of Proposition 1. Let $Z(t,T)$ denote the price of the default-free bond discounted using $B(t)$:

$$Z(t,T) = \frac{P(t,T)}{B(t)}. \tag{A1}$$

Because Z is a martingale under Q for any $t < T$, we must have $Z(t,T) = E^t[Z(t+h,T)]$ or, equivalently,

$$E^t\left[\frac{Z(t+h,T)}{Z(t,T)}\right] = 1. \tag{A2}$$

Now,

$$\frac{Z(t+h,T)}{Z(t,T)} = \left(\frac{P(t+h,T)}{P(t,T)}\right)\left(\frac{B(t)}{B(t+h)}\right). \tag{A3}$$

With the use of Equation 6a and some algebra, we find the first term in Equation A3 to be

$$\frac{P(t+h,T)}{P(t,T)} = \exp\left(-\left\{\sum_{i=(t/h)+1}^{(T/h)-1}[f(t+h,ih)-f(t,ih)]h\right\}+f(t,t)h\right). \tag{A4}$$

The second term in Equation A3, $B(t)/B(t+h)$, is evidently simply $\exp[-f(t,t)h]$. Combining these terms, we obtain

$$\frac{Z(t+h,T)}{Z(t,T)} = \exp\left\{-\sum_{i=(t/h)+1}^{(T/h)-1}[f(t+h,ih)-f(t,ih)]h\right\}. \tag{A5}$$

With Equation A5 used in Equation A2, the martingale condition becomes

$$E^t\left(\exp\left\{-\sum_{i=(t/h)+1}^{(T/h)-1}[f(t+h,ih)-f(t,ih)]h\right\}\right) = 1. \tag{A6}$$

After a substitution for $[f(t+h,ih) - f(t,ih)]$ from Equation 1, Equation A6 is the same as

$$E^t\left(\exp\left\{-\sum_{i=t/h+1}^{T/h-1}[\alpha(t,ih)h^2+\sigma(t,ih)X_0h^{3/2}]\right\}\right) = 1. \tag{A7}$$

Because $\alpha(t,\bullet)$ is known at t, it may be pulled out of the expectation. The result, after some rearranging, is the promised recursive expression relating risk-neutral drift α to volatility σ at each t:

$$\sum_{i=(t/h)+1}^{(T/h)-1}\alpha(t,ih) = \left(\frac{1}{h^2}\right)\ln\left(E^t\left\{\exp\left[-\sum_{i=(t/h)+1}^{(T/h)-1}\sigma(t,ih)X_0h^{3/2}\right]\right\}\right). \tag{A8}$$

Proof of Proposition 2. Pick any $t < T$ and suppose that the time-t rating class of the bond is k. Consider a one-period investment in this bond at t. Then, at time $t+h$, there is a set of possible values $\Pi_j(t+h,T)$, $\forall j = 1, \ldots, K+1$ because the bond may remain in its time-t rating class k or move to any other rating class j. Thus,

$$E_k^t[\Pi(t+h,T)|\text{No default}] = E^t\left[\sum_{j=1}^{K}\frac{q_{k,j}(t)}{1-\lambda_k(t)}\Pi_j(t+h,T)\right]. \tag{A9}$$

The expectation in the right-hand side of Equation A9 is over the state space $(X_0, \mathbf{X})$. Note that $\lambda_k(t) \equiv q_{k,K+1}(t)$ and the sum inside the expectation is over all possible rating classes at $t+h$, conditional on no default at $t+h$.

If the bond has defaulted in the period $(t,t+h]$, a cash flow occurs at $t+h$ because of the recovery upon default. By the RMV (recovery of market value) assumption given in Equation 9, the expected cash flow is $\phi_k(t)E_k^t[\Pi(t+h,T)]$. Because the probability of default by $t+h$, given that the rating class at time t is k, is $\lambda_k(t)$, the undiscounted expected value of the bond is

$$[1-\lambda_k(t)]E_k^t[\Pi(t+h,T)|\text{No default}]+\lambda_k(t)\phi_k(t)E_k^t[\Pi(t+h,T)|\text{No default}],\ \forall k,$$

which is the same as

$$[1-\lambda_k(t)+\lambda_k(t)\phi_k(t)]E_k^t[\Pi(t+h,T)|\text{No default}]\,,\ \forall k.$$

By definition of Q, when discounted at the short rate, $r(t)$, this expected cash flow must equal $\Pi_k(t,T)$, so

$$E^t\left\{\frac{[1-\lambda_k(t)+\lambda_k(t)\phi_k(t)]E_k^t[\Pi(t+h,T)|\text{No default}]}{\exp[r(t)h]\Pi_k(t,T)}\right\} = 1,\ \forall k. \tag{A10}$$

Now, using Equations 10 and 6b and the definitional relationship $s(t,t) = \varphi(t,t) - f(t,t)$, we get

$$\Pi_k(t,T)\exp\left\{\left[r(t)+\sum_{j=1}^{k}s_j(t,t)\right]h\right\} = \exp\left[-\sum_{i=(t/h)+1}^{(T/h)-1}\varphi_k(t,ih)h\right], \tag{A11}$$

and

$$E_k^t[\Pi(t+h,T)|\text{No default}] = E^t\left\{\sum_{j=1}^{K}\frac{q_{k,j}(t)}{1-\lambda_k(t)}\exp\left[-\sum_{i=(t/h)+1}^{(T/h)-1}\varphi_j(t+h,ih)h\right]\right\}. \tag{A12}$$

Using Equations A11 and A12, we get the implicit equation for the drift term, $\beta(t,T)$:

$$E^t\left(\sum_{j=1}^{K}\frac{q_{k,j}(t)}{1-\lambda_k(t)}\exp\left\{-\sum_{i=(t/h)+1}^{(T/h)-1}[\varphi_j(t+h,ih)-\varphi_k(t,ih)]h\right\}\right) = 1. \tag{A13}$$

Now, writing $\varphi_j(t+h,ih) - \varphi_k(t,ih)$ as $\varphi_j(t+h,ih) - \varphi_j(t,ih) + \varphi_j(t,ih) - \varphi_k(t,ih)$ and using Equations 1, 3, and 4, we can rewrite Equation A13 as

$$E^t\left(\begin{array}{c}\displaystyle\sum_{j=1}^{K}\frac{q_{k,j}(t)}{1-\lambda_k(t)}\exp\left\{-\sum_{i=(t/h)+1}^{(T/h)-1}[\alpha(t,ih)h^2+\sigma(t,ih)X_0h^{3/2}]\right\}\\ \displaystyle\times\exp\left\{-\sum_{i=(t/h)+1}^{(T/h)-1}\sum_{l=1}^{j}[\beta_l(t,ih)h^2+\eta_l(t,ih)'\mathbf{X}h^{3/2}]\right\}\\ \displaystyle\times\exp\left\{-\sum_{i=(t/h)+1}^{(T/h)-1}[\varphi_j(t,ih)-\varphi_k(t,ih)]h\right\}\end{array}\right) = 1 \tag{A14}$$

Using the notation $\theta_j(t,ih) = \sum_{l=1}^{j}\beta_l(t,ih)$ and assuming independence of the rating-transition process from the diffusion processes $(X_0, \mathbf{X})$, we get $\forall t$, $\forall k$ at each state, a system of K linear equations in K unknowns (x_j, $j = 1, \ldots, K$):

$$\sum_{j=1}^{K}a_{k,j}b_{k,j}x_j = 1, \tag{A15}$$

where

$$a_{k,j} = \frac{q_{k,j}(t)}{1-\lambda_k(t)}\exp\left[-\sum_{i=(t/h)+1}^{(T/h)-1}\alpha(t,ih)h^2\right]\exp\left\{-\sum_{i=(t/h)+1}^{(T/h)-1}[\varphi_j(t,ih)-\varphi_k(t,ih)]h\right\}, \tag{A16}$$

$$b_{k,j} = E^t\left(\exp\left\{-\sum_{i=(t/h)+1}^{(T/h)-1}\left[\sigma(t,ih)X_0+\sum_{l=1}^{j}\eta_l(t,ih)'\mathbf{X}\right]h^{3/2}\right\}\right), \tag{A17}$$

and

$$x_j = \exp\left[-\sum_{i=(t/h)+1}^{(T/h)-1}\varphi_j(t,ih)h^2\right]. \tag{A18}$$

Notes

1. Reduced-form models are so called to contrast them to "structural models," which build on the work of Merton (1974). Structural models start with a model of the firm-value process and value risky debt by either endogenizing default as a failure of the equityholders to meet the liabilities of the firm or assuming that default is triggered by firm value falling below a threshold barrier.
2. The relationship between the Duffie–Singleton approach and the Das–Sundaram approach is, in a sense, analogous to that between factor models of the term structure (e.g., Vasicek 1977) and the Heath–Jarrow–Morton (1990) approach. In particular, implementation of a factor model requires assumptions about the model's risk premium or, equivalently, about drifts in the risk-neutral world. In contrast, the Heath–Jarrow–Morton model takes the current term structure of the riskless rates and its volatilities as the sole inputs and describes an arbitrage-free evolution of the term structure from this information alone.
3. Other reduced-form models are presented in Duffee (1999), Duffie and Huang (1996), Duffie, Schroder, and Skiadas (1996), Jarrow and Turnbull (1995), and Ramaswamy and Sundaresan (1986).
4. A model that is very close in spirit to ours in continuous time was developed by Bielecki and Rutkowski. This model, also based on HJM, uses information about credit spreads together with information about transition probabilities and recovery rates to develop a conditionally Markovian model of credit risk. Bielecki and Rutkowski modeled the evolution of spreads of each rating class directly, however, not via inter-rating spreads as we do.
5. Specifically, we assume that Q is an equivalent martingale measure with respect to the money market account $B(t)$ defined in Equation 13. See Harrison and Kreps (1979) or Harrison and Pliska (1980) for the role of equivalent martingale measures in securities modeling.
6. The linear equations are solved by using standard algorithms, such as Gauss–Seidel (see Press, Teukofsky, Vetterling, and Flannery 1992).
7. Where w.p. = with probability.
8. For some numerical estimates of the correlation coefficient between corporate spreads and interest rates in general, see Das and Sundaram or Das and Tufano.
9. The code for calibrating the tree is available at http://www.stern.nyu.edu/~rsundara/publns.htm in the paper "Arbitrage-Free Pricing of Credit Derivatives with Ratings Transitions." This computer implementation uses a recursive scheme that is convenient and seamlessly processes the forward induction and backward recursion needed to compute more complicated derivative securities.
10. Another, more recent, example of a CSN is an issue by Olivetti, which announced on June 7, 2000, that it planned to link the coupon on €18 billion ($17 billion) of bonds sold by itself and its Tecnost SpA unit to the bonds' credit rating. Investors are to be paid more if the rating worsens and paid less if the grade recovers. As stated by Olivetti Chief Financial Officer Luciano La Noce, "The coupon adjustment will be applicable to all of the outstanding issues. Going forward, we think having these sort of volatility protection measures associated with our bonds should result in a lower capital cost" (Bloomberg Equity News, June 16, 2000).

References

Adler, D., and E. Altman. 1998. "The Anatomy of the High Yield Bond Market." Global Corporate Bond Research, Salomon Smith Barney.

Altman, E., and V. Kishore. 1996. "Almost Everything You Wanted to Know about Recoveries on Defaulted Bonds." *Financial Analysts Journal*, vol. 52, no. 6 (November/December):57–63.

Arvantis, A., J. Gregory, and J-P. Laurent. 1999. "Building Models for Credit Spreads." *Journal of Derivatives*, vol. 6, no. 3 (Spring):27-43.

Bhattacharya, S., and S. Mason. 1981. "Risky Debt, Jump Processes and Safety Covenants." *Journal of Financial Economics*, vol. 9, no. 3:281–307.

Bielecki, T.R., and M. Rutkowski. 2000. "Multiple Ratings Model of Defaultable Term Structure." *Mathematical Finance*, vol. 10, no. 2 (April):125–139.

Black, F., and J. Cox. 1976. "Valuing Corporate Securities: Some Effects of Bond Indenture Provisions." *Journal of Finance*, vol. 31, no. 2 (May):361–367.

Black, F., and M. Scholes. 1973. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy*, vol. 81, no. 3 (May/June):637–654.

Black, F., E. Derman, and W. Toy. 1990. "A One-Factor Model of Interest Rates and Its Application to Treasury Bond Options." *Financial Analysts Journal*, vol. 46, no. 1 (January/February):33–39.

Crosbie, P. 1997. "Modeling Default Risk." KMV Corporation.

Das, S.R. 1995. "Credit Risk Derivatives." *Journal of Derivatives*, vol. 2, no. 3 (Spring):7–23.

Das, S.R. 1999. "Pricing Credit Derivatives." In *Handbook of Credit Derivatives*. Edited by J. Francis, J. Frost, and J.G. Whittaker. New York: McGraw-Hill.

Das, S.R., and R. Sundaram. 2000. "A Discrete-Time Approach to Arbitrage-Free Pricing of Credit Derivatives." *Management Science*, vol. 46, no. 1 (January):46–62.

Das, S.R., and P. Tufano. 1996. "Pricing Credit Sensitive Debt When Interest Rates, Credit Ratings, and Credit Spreads Are Stochastic." *Journal of Financial Engineering*, vol. 5, no. 2 (June):161–198.

Delianedis, G., and R. Geske. 1998. "Credit Risk and Risk Neutral Default Probabilities: Information about Rating Migrations and Defaults." Working paper, University of California, Los Angeles.

Duffee, G. 1998. "The Relation between Treasury Yields and Corporate Bond Yield Spreads." *Journal of Finance*, vol. 53, no. 6 (December):2225–42.

———. 1999. "Estimating the Price of Default Risk." *Review of Financial Studies*, vol. 12, no. 1 (January):197–226.

Duffie, D. 1999. "Credit Swap Valuation." *Financial Analysts Journal*, vol. 55, no. 1 (January/February):73–87.

Duffie, D., and M. Huang. 1996. "Swap Rates and Credit Quality." *Journal of Finance*, vol. 51, no. 3 (July):921–950.

Duffie, D., and K. Singleton. 1999. "Modeling Term Structures of Defaultable Bonds." *Review of Financial Studies*, vol. 12, no. 4 (October):687–720.

Duffie, D., M. Schroder, and C. Skiadas. 1996. "Recursive Valuation of Defaultable Securities and the Timing of Resolution of Uncertainty." *Annals of Applied Probability*, vol. 6, no. 4 (November):1075–90.

Geske, R. 1977. "Valuation of Corporate Liabilities as Compound Options." *Journal of Financial and Quantitative Analysis*, vol. 12, no. 4 (November):541–552.

Harrison, J.M., and D.M. Kreps. 1979. "Martingales and Arbitrage in Multi-Period Securities Markets." *Journal of Economic Theory*, vol. 20, no. 3 (June):381–408.

Harrison, J.M., and S.R. Pliska. 1980. "Martingales and Stochastic Integrals in the Theory of Continuous Trading." *Stochastic Processes and Their Applications*, vol. 11 (January):215–260.

Heath, D., R.A. Jarrow, and A. Morton. 1990. "Bond Pricing and the Term Structure of Interest Rates: A Discrete Time Approximation." *Journal of Financial and Quantitative Analysis*, vol. 25, no. 4 (December):419–440.

Huang, J. 1996. "Option to Default and Optimal Debt Service." Working paper, New York University.

Jarrow, R., and S. Turnbull. 1995. "Pricing Derivatives on Financial Securities Subject to Credit Risk." *Journal of Finance*, vol. 50, no. 1 (March):53–86.

Jarrow, R., D. Lando, and S. Turnbull. 1997. "A Markov Model for the Term Structure of Credit Spreads." *Review of Financial Studies*, vol. 10, no. 2 (April):481–523.

Kijima, M. 1998. "Monotonicities in a Markov Chain Model for Valuing Corporate Bonds Subject to Credit Risk." *Mathematical Finance*, vol. 8, no. 3 (July):229–247.

Kijima, M., and K. Komoribayashi. 1998. "A Markov Chain Model for Valuing Credit Derivatives." *Journal of Derivatives*, vol. 6, no. 1 (Fall):97–108.

Kim, I.J., K. Ramaswamy, and S. Sundaresan. 1989. "The Valuation of Corporate Fixed Income Securities." Working paper, Wharton School, University of Pennsylvania.

———. 1993. "Does Default Risk in Coupons Affect the Valuation of Corporate Bonds?: A Contingent Claims Model." *Financial Management*, vol. 22, no. 3 (Autumn):117–131.

KMV Corporation. 1993. *Credit Monitor Overview*. San Francisco, CA.

Lando, D. 1998. "Cox Processes and Credit-Risky Bonds." *Review of Derivatives Research*, vol. 2, no. 2/3:99–120.

Leland, H. 1994. "Corporate Debt Value, Bond Covenants, and Optimal Capital Structure." *Journal of Finance*, vol. 49, no. 4 (September):1213–52.

Longstaff, F., and E. Schwartz. 1995. "A Simple Approach to Valuing Risky Fixed and Floating Rate Debt." *Journal of Finance*, vol. 50, no. 3 (July):789–819.

Madan, D., and H. Unal. 1998. "Pricing the Risks of Default." *Review of Derivatives Research*, vol. 2, no. 2/3:121–160.

Madan, D., and H. Unal. 2000. "A Two Factor Hazard Rate Model for Pricing Risky Debt in a Complex Capital Structure." *Journal of Financial and Quantitative Analysis*, vol. 35, no. 1 (March):43–65.

Merton, R.C. 1974. "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates." *Journal of Finance*, vol. 29, no. 2 (May):449–470.

Neilsen, L., J. Saa-Requejo, and P. Santa-Clara. 1993. "Default Risk and Interest Rate Risk: The Term Structure of Default Spreads." Working paper, INSEAD, Fontainbleau, France.

Press, W.H., S.A. Teukofsky, W.T. Vetterling, and B.P. Flannery. 1992. *Numerical Recipes in C*. Second ed. Cambridge, U.K.: Cambridge University Press.

Ramaswamy, K., and S. Sundaresan. 1986. "The Valuation of Floating Rate Instruments: Theory and Evidence." *Journal of Financial Economics*, vol. 17, no. 2 (December):251–272.

Schönbucher, P.J. 1998. "Term Structure Modeling of Defaultable Bonds." *Review of Derivatives Research*, vol. 2, no. 2/3:161–192.

Shimko, D., N. Tejima, and D. Van-Deventer. 1993. "The Pricing of Risky Debt When Interest Rates Are Stochastic." *Journal of Fixed Income*, vol. 3, no. 2 (September):58–65.

Skora, R. 1998. "Credit Modeling and Credit Derivatives: Rational Modeling." Working paper, Skora and Company.

Vasicek, O. 1977. "An Equilibrium Characterization of the Term Structure." *Journal of Financial Economics*, vol. 5, no. 2 (November):177–188.

Wilson, T. 1997. "Portfolio Credit Risk (I)." *Risk*, vol. 10, no. 9 (September):111–117.

Wolfram, S. 1988. *Mathematica: A System for Doing Mathematics by Computer*. Redwood City, CA: Addison-Wesley.