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Systemic Risk and International Portfolio Choice

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ABSTRACT

Returns on international equities are characterized by jumps; moreover, these jumps tend to occur at the same time across countries leading to *systemic risk*. We capture these stylized facts using a multivariate system of jump-diffusion processes where the arrival of jumps is simultaneous across assets. We then determine an investor's optimal portfolio for this model of returns. Systemic risk has two effects: One, it reduces the gains from diversification and two, it penalizes investors for holding levered positions. We find that the loss resulting from diminished diversification is small, while that from holding very highly levered positions is large.

RETURNS ON INTERNATIONAL EQUITIES are characterized by jumps;[1] moreover, these jumps tend to occur at the same time across countries, implying that conditional correlations between international equity returns tend to be higher in periods of high market volatility or following large downside moves.[2] Our objective in

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[1] Evidence on jumps in international equity returns is provided by Jorion (1988), Akgiray and Booth (1988), Bates (1996), and Bekaert et al. (1998).

[2] For example, on July 19, 2002, the Dow fell by 4.6%, the Dax by 5.0%, the Cac by 5.4%, the FTSE by 4.6%, and the Nikkei by 2.8%. Similarly, world equity markets fell in lockstep on October 27, 1997, when the drop from the 12-month peak was 9.2% in Britain, 35.4% in Hong Kong, 21.3% in Japan, 2.1% in Australia, 10.7% in Mexico, 27.9% in Brazil, and 9.1% in the United States. Other events with large correlated price drops include the Debt crisis of 1982, the Mexican crisis in December 1994, and the Russian crisis in August 1998; see Rigobon (2003) for a complete list of dates with large market moves. For evidence on changing conditional correlations see, for instance, Speidell and Sappenfield (1992), Odier and Solnik (1993), Erb, Harvey, and Viskanta (1994), Longin and Solnik (1995), Karolyi and Stulz (1996), Chakrabarti and Roll (2000), and Ang and Chen (2002).

this paper is to evaluate the effect on portfolio choice of *systemic risk*, defined as the risk from infrequent events that are highly correlated across a large number of assets.

Our contribution is to provide a mathematical model of security returns that captures the stylized facts about international equity returns described above. We do this by modeling security returns as jump-diffusion processes where jumps across assets are systemic (occur simultaneously), though the size of each jump is allowed to differ across assets. Next, we derive the optimal portfolio weights for this model of returns. Then, we calibrate the portfolio model to the U.S. equity index and to five international equity indexes. For robustness, we consider two sets of international indexes: the first for developed countries and the second for emerging countries. Systemic risk has two effects: It reduces the gains from diversification and also penalizes investors for holding levered positions. We find that the loss resulting from diminished diversification is small, while that from holding highly levered positions is large. For instance, the certainty equivalent cost of ignoring systemic risk for a conservative agent with relative risk aversion of 3 who is investing \$1,000 for 1 year is on the order of \$0.10 for the developed-country indexes and \$3 for the emerging-country indexes. However, for more aggressive investors who hold heavily levered portfolios, the cost of ignoring systemic risk is substantial: For example, an investor with a risk aversion of 1 who ignores systemic risk has a positive probability of losing all her wealth.

Our work can be distinguished from the literature on portfolio choice with *idiosyncratic* jumps in returns, for example, Aase (1984), Jeanblanc-Picque and Pontier (1990), and Shirakawa (1990). In more recent work, Liu, Longstaff, and Pan (2003) study a model of portfolio choice with event risk. In contrast to these theoretical models, our motivation is to evaluate the effect of *systemic* jumps on portfolio selection by empirically estimating the parameters of the returns in our model, and implementing the model based on these estimates. In contrast to the *static* model in Chunhachinda et al. (1997), where polynomial goal programming is used to examine the effect of skewness on portfolio choice by assuming a utility function defined over the moments of the distribution of returns, our model is dynamic, with preferences given by a standard constant relative risk-averse utility function, and in our model the effect of skewness (and higher moments) arises because of jumps in the returns process rather than being introduced explicitly through the utility function.[3]

Our work is also related to Ang and Bekaert (2002), who embed an international portfolio choice problem in a dynamic model with a regime-switching data-generating process. Two regimes are considered that correspond to a

[3] For early work on how skewness influences portfolio choice, see Samuelson (1970), Tsiang (1972), and Kane (1982). Kraus and Litzenberger (1976) show the implications for equilibrium prices of a preference for positive skewness, while Kraus and Litzenberger (1983) derive the sufficient conditions on return distributions to get a three-moment (mean, variance, and skewness) capital asset pricing model. Harvey and Siddique (2000) provide an empirical test of the effect of skewness on asset prices.

normal regime with low correlations and a downturn regime with higher correlations. In their setup, regimes can be persistent, and their paper includes an analysis of portfolio choice when the short interest rate and earnings yields predict returns. The framework in Ang and Bekaert, however, does not accommodate intermediate consumption, admits only a numerical solution even in the absence of intermediate consumption, and is difficult to estimate when there are more than two regimes or three risky assets.

In contrast to Ang and Bekaert, we develop a theoretical framework along the lines of Merton (1971); because our model nests the well-understood Merton model as a special case, it allows one to interpret cleanly the effect of systemic jumps. Also, we provide analytic expressions for the optimal portfolio weights. Moreover, our model can incorporate intermediate consumption (the solution to the portfolio problem stays the same), and can be estimated and implemented for any number of assets. While in the paper we consider only a simple IID environment where the unconditional correlations between returns are constant over time, we argue that this is sufficient to show that the effect of systemic jumps will not be large even in the presence of regime shifts.[4] Our framework can also allow for predictability in returns and for other state variables, just as they are incorporated in Merton, but because this is tangential to our main objective, we do not include these features in the model we present.

The rest of the paper is organized as follows. In Section I, we develop a model of asset returns that captures systemic risk. In Section II, we derive the optimal portfolio weights when asset returns have a systemic-jump component. In Section III, we describe our data, give the moments for the returns in the presence of systemic risk, and estimate the parameters of the returns processes. We then calibrate the portfolio model to the estimated parameters in order to compare the portfolio weights of an investor who accounts for systemic risk and an investor who ignores systemic risk. We conclude in Section IV. Proofs for propositions are presented in the Appendix.

I. Asset Returns with Systemic Risk

In this section, we develop a model of asset prices that allows for systemic jumps and compare it to a pure-diffusion model without jumps. The two features of the data that we wish our returns-model to capture are (1) large changes in asset prices, and (2) a high degree of correlation across these changes. To allow for large changes in returns, we introduce a jump component in prices; to model these jumps as being systemic, we assume that this jump is common across all assets, though the distribution of the jump size is allowed to vary across assets.

We start by describing the standard continuous-time process that is typically assumed for asset returns:

$$\frac{dS_n}{S_n} = \hat{\alpha}_n\, dt + \hat{\sigma}_n\, dz_n, \quad n = 1, \ldots, N, \tag{1}$$

[4] Das and Uppal (2003) show how to extend the model to allow for persistence in jumps.

with

$$E_t\left[\frac{dS_n}{S_n}\right] = \hat{\alpha}_n\,dt \tag{2}$$

$$E_t\left[\left(\frac{dS_n}{S_n}\right)\times\left(\frac{dS_m}{S_m}\right)\right] = \hat{\sigma}_{nm}\,dt = \hat{\sigma}_n\hat{\sigma}_m\hat{\rho}_{nm}\,dt, \tag{3}$$

where S_n is the price of asset n, N is the total number of risky assets being considered for the portfolio, and the correlation between the shocks dz_n and dz_m is denoted by $\hat{\rho}_{nm}\,dt = E(dz_n \times dz_m)$. We will denote the $N \times N$ matrix of the covariance terms arising from the diffusion components by $\widehat{\boldsymbol{\Sigma}}$, with its typical element being $\hat{\sigma}_{nm} \equiv \hat{\sigma}_n\hat{\sigma}_m\hat{\rho}_{nm}$. We adopt the convention of denoting vectors and matrices with boldface characters in order to distinguish them from scalar quantities; parameters of the pure-diffusion returns process, and other quantities related to the pure-diffusion model, are denoted with a ˆ (carat) over the variable.

To allow for the possibility of infrequent but large changes in asset returns,[5] we extend the specification in equation (1) by introducing a jump component to the process for returns, as in Merton (1976). In order to capture the systemic nature of these jumps, we impose two restrictions on the jump-diffusion processes: one, the jump is assumed to arrive at the same time across all risky assets; two, conditional on a jump, the jump size is assumed to be perfectly correlated across assets; that is, the value of all the assets jumps in the same direction. Below, we formally describe a returns process for the risky assets that has these properties.

Introducing a jump component to the process of returns in (1), we have

$$\frac{dS_n}{S_n} = \alpha_n\,dt + \sigma_n\,dz_n + (\tilde{J}_n - 1)\,dQ(\lambda), \quad n = 1,\ldots,N, \tag{4}$$

where Q is a Poisson process with intensity λ, and $(\tilde{J}_n - 1)$ is the random jump amplitude that determines the percentage change in the asset price if the Poisson event occurs. Given our desire to model the large changes in prices as occurring at the same time across the risky assets, we have assumed that the arrival of jumps is coincident across all risky assets; that is, $dQ_n(\lambda_n) = dQ_m(\lambda_m) = dQ(\lambda), \forall n = \{1,\ldots,N\}, m = \{1,\ldots,N\}$.[6] We also assume that the diffusion shock, the Poisson jump, and the random variable $\tilde{J}_n$ are independent and that $J_n \equiv \ln(\tilde{J}_n)$ has a normal distribution with mean μ_n and variance ν_n^2, implying that the distribution of the jump size is asset-specific (below we will assume

[5] In contrast to systemic risk, *systematic* risk refers to correlation between assets and a common factor, but does not require that the size of this correlation be large or that the correlated changes be infrequent.

[6] The returns process described above is IID; in particular, and in contrast to Ang and Bekaert (2002), there is no persistence in jumps. Das and Uppal (2003) show how one could extend this model to allow for persistence in systemic jumps by making the arrival rate of jumps, λ, stochastic.

that, conditional on a jump, the jump sizes for different assets are perfectly correlated).

Thus, for the process in (4), the total expected return in equation (5) has two components: One part comes from the diffusion process, α_n and the other, denoted α_n^{J}, comes from the jump process:

$$E_t\left[\frac{dS_n}{S_n}\right] = \alpha_n\,dt + \alpha_n^{\mathsf{J}}\,dt. \tag{5}$$

We also assume that the jump size is perfectly correlated across assets; as we shall see in Section III, this turns out to be a conservative assumption, and has the further advantage of reducing the number of parameters to be estimated. The total covariance between dS_n and dS_m, given in the equation below,

$$E_t\left[\left(\frac{dS_n}{S_n}\right)\times\left(\frac{dS_m}{S_m}\right)\right] = \sigma_{nm}\,dt + \sigma_{nm}^{\mathsf{J}}\,dt, \tag{6}$$

arises from two sources: The covariance between the diffusion components of the returns, $\sigma_{nm} \equiv \sigma_n\sigma_m\rho_{nm}$, and the covariance between the jump components, σ_{nm}^{J}. The $N\times N$ matrix containing the covariation arising from the jump terms is denoted by Σ^{J}, while the $N\times N$ matrix of the covariance terms arising from the diffusion components is denoted by Σ, with its typical element being $\sigma_{nm} \equiv \sigma_n\sigma_m\rho_{nm}$. Explicit expressions for α_n^{J} in (5) and σ_{nm}^{J} in (6), in terms of the parameters of the underlying returns processes, $\{\lambda, \mu_n, \nu_n\}$, are given in equations (26) and (27).

In our experiment, we wish to compare the portfolio of an investor who models security returns using the pure-diffusion process in (1) with that of an investor who accounts for systemic risk by using the jump-diffusion process in (4) but matches the first two moments of returns. Thus, we need to choose the parameters of the jump-diffusion processes in such a way that the first two moments for this process given in equations (5) and (6) match exactly the first two moments of the pure-diffusion returns process in equations (2) and (3).[7] Even though it is straightforward to do this, we highlight it in a proposition because this result is important for understanding our analysis.

PROPOSITION 1: *In order for the first and second moments from the jump-diffusion process to match the corresponding moments from the pure-diffusion process, we set, for $n, m = \{1,\ldots,N\}$,*

$$\alpha_n = \hat{\alpha}_n - \alpha_n^{\mathsf{J}}, \tag{7}$$

$$\sigma_{nm} = \hat{\sigma}_{nm} - \sigma_{nm}^{\mathsf{J}}. \tag{8}$$

One interpretation of the above *compensation* of the parameters is that the investor using the jump-diffusion returns process takes the total expected

[7] An important difference between our work and that of Liu et al. (2003) is that while we control for the magnitude of the first two moments when comparing the portfolio strategy that accounts for jumps with the one that ignores jumps, they do not.

return on the asset, $\hat{\alpha}_n$, and the covariance, $\hat{\sigma}_{nm}$, and subtracts from them α_n^{J} and σ_{nm}^{J}, respectively, with the understanding that this will be added back through the jump term, $(\tilde{J}_n - 1)\,dQ(\lambda)$. In this way, she reduces the expected return and covariance coming from the diffusion terms in order to offset exactly the contribution of the jump.

Even though the unconditional expected return and covariance under the compensated jump-diffusion process will match those from the pure-diffusion process, the two processes will not lead to identical portfolios. This is because the jump also introduces skewness and kurtosis into the returns process (see equations (24) and (25)).[8] In the next section, we analyze the difference between the portfolio of an investor who allows for systemic jumps in returns and an investor who ignores this effect.

II. Portfolio Selection in the Presence of Systemic Risk

In this section, we formulate and solve the portfolio selection problem when returns are given by the jump-diffusion process in (4). Given that financial markets are incomplete in the presence of jumps of random size, we determine the optimal portfolio weights using stochastic dynamic programming rather than the martingale pricing approach.

Our modeling choices are driven by the desire to develop the simplest possible framework in which one can examine the portfolio selection problem in the presence of systemic risk. Hence, we work with a model that has a constant investment opportunity set; an extension of this model to the case where the investment opportunity set is changing over time, via shifts in the likelihood of systemic jumps, is considered in Das and Uppal (2003). Also, we model the portfolio problem in continuous time because of the analytical convenience this affords. Finally, we describe the model in the context of international portfolio selection, but the model applies to any set of securities with appropriate returns processes.

A. Optimal Portfolio Weights

We consider a U.S. investor who wishes to maximize the expected utility from terminal wealth,[9] W_T, with utility being given by $U(W_T) = \frac{W_T^{1-\gamma}}{1-\gamma}$, where $\gamma > 0$, $\gamma \neq 1$, so that constant relative risk aversion is equal to γ.[10] The investor can allocate funds across $n = \{0, 1, \ldots, N\}$ assets: a riskless asset denominated in U.S. dollars ($n = 0$), a risky U.S. equity index ($n = 1$), and risky foreign equity indexes, $n = \{2, \ldots, N\}$.

[8] Jumps are not the only way of introducing skewness and kurtosis into the process for returns—stochastic volatility would also generate such effects. Of course, jumps have the additional effect that they constrain portfolio weights in order to prevent wealth from becoming negative.

[9] We do not consider intermediate consumption because it has no effect on the optimal weights in our model.

[10] For the case where $\gamma = 1$, the utility function is given by $U(W_T) = \ln W_T$.

The price process for the riskless asset, S_0, is

$$dS_0 = rS_0\,dt, \tag{9}$$

where r is the instantaneous riskless rate of interest, which is assumed to be constant over time. The stochastic process for the price of each equity index (in dollar terms)[11] with a common jump term is as given in equation (4), which is restated below:

$$\frac{dS_n}{S_n} = \alpha_n\,dt + \sigma_n\,dz_n + (\tilde{J}_n - 1)\,dQ(\lambda), \quad n = 1, \ldots, N, \tag{10}$$

with α_n and σ_{nm} defined in equations (7) and (8).

Denoting the proportion of wealth invested in asset n by $w_n, n = \{1, \ldots, N\}$, the investor's problem at t can be written as

$$V(W_t, t) \equiv \max_{\{w_n\}} E\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right], \tag{11}$$

subject to the dynamics of wealth

$$\frac{dW_t}{W_t} = [\mathbf{w}'\mathbf{R} + r]\,dt + \mathbf{w}'(\boldsymbol{\sigma}\cdot\mathbf{dZ}_t) + \mathbf{w}'\mathbf{J}_t\,dQ(\lambda), \quad W_0 = 1, \tag{12}$$

where $\mathbf{w}$ is the $N \times 1$ vector of risky-asset portfolio weights, $\mathbf{R} \equiv \{\alpha_1 - r, \ldots, \alpha_N - r\}'$ is the excess-returns vector, σ is the vector of volatilities, $\mathbf{dZ}$ is the vector of diffusion shocks, with the dot product $\boldsymbol{\sigma}\cdot\mathbf{dZ}_t$ denoting element-by-element multiplication of σ_n and dZ_n, and $\mathbf{J}_t \equiv [\tilde{J}_1 - 1, \tilde{J}_2 - 1, \ldots, \tilde{J}_N - 1]'$ is the vector of random jump amplitudes for the N assets at time t. The covariance matrix of the diffusion component of the joint stochastic process is given by $\boldsymbol{\Sigma}$.

Using the standard approach to stochastic dynamic programming and the appropriate form of Ito's lemma for jump-diffusion processes, one can obtain the following Hamilton–Jacobi–Bellman equation

$$\begin{aligned}0 = \max_{\{\mathbf{w}\}}\Bigg\{&\frac{\partial V(W_t,t)}{\partial t} + \frac{\partial V(W_t,t)}{\partial W}W_t[\mathbf{w}'\mathbf{R} + r] + \frac{1}{2}\frac{\partial^2 V(W_t,t)}{\partial W^2}W_t^2\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}\\ &+ \lambda E[V(W_t + W_t\mathbf{w}'\mathbf{J}_t, t) - V(W_t,t)]\Bigg\},\end{aligned} \tag{13}$$

where the terms on the first line are the standard terms when the processes for returns are continuous, and the term on the second line accounts for jumps in returns.

[11] The dollar return on a foreign equity index includes the return on currency and the return on the international equity index in local-currency terms. For international equity returns, one could model separately the equity return in local-currency terms and the return on currency. We do not do this because it complicates the notation without adding any insights.

We guess (and verify) that the solution to the value function is of the following form:

$$V(W_t, t) = A(t)\frac{W_t^{1-\gamma}}{1-\gamma}. \tag{14}$$

Expressing the jump term using this guess for the value function (details are in the proof for the proposition below), and simplifying the resulting differential equation, we get an equation that is independent of wealth:

$$0 = \max_{\{\mathbf{w}\}}\left\{\frac{1}{A(t)}\frac{dA(t)}{dt} + (1-\gamma)[\mathbf{w}'\mathbf{R} + r] - \frac{(1-\gamma)\gamma}{2}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} + \lambda E\big[(1+\mathbf{w}'\mathbf{J}_t)^{1-\gamma} - 1\big]\right\}. \tag{15}$$

Differentiating the above with respect to $\mathbf{w}$, one gets the following result.

PROPOSITION 2: *The optimal portfolio weights in the presence of systemic risk are given by the solution to the following system of N nonlinear equations:*

$$\mathbf{0} = \mathbf{R} - \gamma\boldsymbol{\Sigma}\mathbf{w} + \lambda E\big[\mathbf{J}_t(1+\mathbf{w}'\mathbf{J}_t)^{-\gamma}\big], \quad \forall t. \tag{16}$$

Note that (16) gives only an implicit equation for the unconditional portfolio weights, $\mathbf{w}$. Thus, to determine the magnitude of the optimal portfolio weights, one needs to solve this equation numerically, which we do in Section III.

In contrast to the above solution, an investor who ignores the possibility of systemic jumps and assumes the standard model in which price processes are multivariate diffusions without jumps will choose the portfolio weights given by the familiar Merton (1971) expression below.

COROLLARY 1: *The weights chosen by an investor who assumes that returns are described by the pure-diffusion process in equation (1) are*

$$\hat{\mathbf{w}} = \frac{1}{\gamma}\widehat{\boldsymbol{\Sigma}}^{-1}\hat{\mathbf{R}}. \tag{17}$$

The difference between the portfolio of the investor who accounts for systemic jumps, $\mathbf{w}$, and that of an investor who ignores this feature of the data and chooses portfolio $\hat{\mathbf{w}}$ can be understood by comparing equation (16) and (17): The two equations are the same when there are no jumps ($\lambda = 0$). Thus, the difference between $\mathbf{w}$ and $\hat{\mathbf{w}}$ arises from the higher moments ignored in (17).[12]

[12] This also shows how our model is related to that of Chunhachinda et al. (1997), who use polynomial goal programming in a single-period model to examine the effect of skewness on portfolio choice by assuming a utility function defined over the moments of the distribution of returns. In contrast, we work with the standard power utility function that is commonly used to examine optimal portfolio selection and instead modify the returns process to allow for the possibility of skewness and higher moments.

B. Certainty Equivalent Cost of Ignoring Systemic Risk

Above, we have compared the optimal portfolio weights for an investor who accounts for systemic jumps in returns and the investor who ignores this feature of the data. In this section, we compare the certainty equivalent cost of following the suboptimal portfolio strategy. The objective of this exercise is to express in dollar terms the cost of ignoring systemic risk.

In order to quantify the cost of ignoring systemic jumps, we compute the additional wealth needed to raise the expected utility of terminal wealth under the suboptimal portfolio strategy to that under the optimal strategy. In this comparison, we denote by CEQ the additional wealth that makes lifetime expected utility under $\hat{\mathbf{w}}$, the portfolio policy that ignores systemic risk, equal to that under the optimal policy, $\mathbf{w}$. Using the notation $V(W_t, t; \mathbf{w}_i)$, where $\mathbf{w}_i = \{\mathbf{w}, \hat{\mathbf{w}}\}$ indicates the particular portfolio weights used to compute the value function, the compensating wealth, CEQ, is computed as follows:

$$V((1 + \mathrm{CEQ})W_t, t; \hat{\mathbf{w}}) = V(W_t, t; \mathbf{w}). \tag{18}$$

Then, from equations (14) and (18), we have

$$A(t; \hat{\mathbf{w}})\left[\frac{1}{1-\gamma}((1 + \mathrm{CEQ})W_t)^{1-\gamma}\right] = A(t; \mathbf{w})\left[\frac{1}{1-\gamma}W_t^{1-\gamma}\right], \tag{19}$$

which implies that

$$\mathrm{CEQ} = \left[\frac{A(t; \mathbf{w})}{A(t; \hat{\mathbf{w}})}\right]^{1/(1-\gamma)} - 1, \tag{20}$$

where, from the proof for Proposition 2,

$$A(t; \mathbf{w}_i) \equiv e^{((1-\gamma)[\mathbf{w}_i'\mathbf{R}+r]-\frac{1}{2}\gamma(1-\gamma)\mathbf{w}_i'\boldsymbol{\Sigma}\mathbf{w}_i+\lambda E[(1+\mathbf{w}_i'\mathbf{J}_t)^{1-\gamma}-1])(T-t)}, \tag{21}$$

with $\mathbf{w}_i = \{\mathbf{w}, \hat{\mathbf{w}}\}$.

III. Calibrating the Effect of Systemic Risk

In this section, we evaluate the effect of systemic risk on portfolio choice by calibrating the jump-diffusion model to returns on the U.S. equity index, and to five international equity indexes. This section is divided into three subsections. In the first, we describe the data and explain how we use the method of moments to estimate the parameters of the returns processes. In the second, we evaluate the effect of systemic risk on portfolio policies using the estimated values for the parameters of the returns process. In the third subsection, we evaluate the sensitivity of our results to the choices we have made in undertaking the calibration exercise.

A. Description of the Data and Estimation of the Model

The data for the developed countries consist of the month-end U.S. dollar values of the equity indexes for the period January 1982 to February 1997 for the United States (U.S.), United Kingdom (U.K.), Switzerland (SW), Germany (GE), France (FR), and Japan (JP). The data for emerging economies are for the period January 1980 to December 1998, and consist of the beginning-of-month value of the equity index for the United States (USA), Argentina (ARG), Hong Kong (HKG), Mexico (MEX), Singapore (SNG), and Thailand (THA). To distinguish the two sets of data, we abbreviate the countries in the developed-economy data set with two characters and denote countries in the emerging-economy data set with three characters.

Table I reports the descriptive statistics for the continuously compounded monthly return on index j in U.S. dollars, r_{jt}, which is defined as the ratio of the log of the index value at time t and its lagged value: $r_{jt} = \ln\left[\frac{S_{jt}}{S_{j,t-1}}\right]$, where S_{jt} is the U.S. dollar value of the index at time t. Examining first the moments for developed economies, we observe from Panel A of Table I that the excess kurtosis

Table I

Descriptive Statistics for Equity Returns—Univariate

Panel A of this table gives the first four moments of the monthly returns in U.S. dollar terms for the developed-country indexes and Panel B gives the same information for the U.S. and five emerging markets. The data for the developed countries are for the period January 1982 to February 1997, and include 182 observations of month-end values of the equity indexes for the United States (U.S.), United Kingdom (U.K.), Japan (JP), Germany (GE), Switzerland (SW), and France (FR). The data for emerging economies consist of 227 observations of the beginning-of-month value of the equity indexes for the USA, Argentina (ARG), Hong Kong (HKG), Mexico (MEX), Singapore (SNG), and Thailand (THA) for the period January 1980 to December 1998.

Panel A: Developed Countries							
	U.S.	U.K.	JP	GE	SW	FR	Avg.
Mean	0.0102	0.0084	0.0080	0.0120	0.0102	0.0113	0.0100
Standard deviation	0.0420	0.0567	0.0697	0.0576	0.0515	0.0611	0.0564
Skewness	−1.1648	−0.4623	−0.0508	−0.2308	−0.6382	−0.4325	−0.4966
Significance level	0.0000	0.0115	0.7815	0.2074	0.0004	0.0181	
Excess kurtosis	7.2236	1.9212	0.8754	2.9546	5.5405	1.5780	3.3489
Significance level	0.0000	0.0000	0.0180	0.0000	0.0000	0.0000	
Panel B: Emerging Countries							
	USA	ARG	HKG	MEX	SNG	THA	Avg.
Mean	0.0104	0.0040	0.0076	0.0034	0.0065	0.00004	0.0053
Standard deviation	0.0414	0.2153	0.1026	0.1437	0.0772	0.1037	0.1140
Skewness	−1.1353	0.1187	−1.4163	−2.0224	−0.7684	−0.6077	−0.9719
Significance level	0.0000	0.4681	0.0000	0.0000	0.0000	0.0002	
Excess kurtosis	6.1823	6.2377	6.9388	9.1851	4.8603	3.7800	6.1974
Significance level	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	

of returns is substantially greater than that for normal distributions (in the table, we report kurtosis in excess of 3, which is the kurtosis for the normal distribution). The excess kurtosis in the data ranges from 0.87 for France to 7.22 for the U.S. For the data on emerging economies, as one would expect, the excess kurtosis is much greater, ranging from 3.77 for Thailand to 9.18 for Mexico. All 12 kurtosis estimates are significant. There are two possible reasons for the kurtosis: (1) When the multivariate return series is not stationary, the mixture of distributions results in kurtosis; and (2), if the returns are characterized by large shocks, then the outliers inject kurtosis.

The second feature of the data is that the skewness of returns for all the developed-market indexes is negative, and for the emerging-country indexes it is more strongly negative, except for Argentina, where it is insignificantly different from zero. The negative skewness is a well-known feature of equity index time series over this time period (1982–1997). Within this period, there were several large negative shocks to the markets contributing to the negative skewness: for instance, the market crash of October 1987, the outbreak of the Gulf War in August 1990, the Mexican crisis in December 1994, and the Russian crisis in August 1998.[13]

Table II reports the covariances and correlations between the returns on the international equity indexes. The correlations for the developed countries range from a low of 0.33 between the United States and Japan, to a high of 0.68 between Germany and Switzerland. The average correlation between the equity markets for developed countries is 0.51. For the emerging countries, the correlations range from the very low 0.05 between Hong Kong and Argentina to 0.55 between Singapore and the United States. The average correlation for the emerging countries is 0.31, which, as one would expect, is much lower than that for the developed countries.

For the benchmark case of the pure-diffusion process in equation (1), the parameters to be estimated are $\{\hat{\alpha}, \widehat{\Sigma}\}$, with the moment conditions available being the ones in equations (2) and (3). From these moment conditions we see that $\{\hat{\alpha}, \widehat{\Sigma}\}$ can be estimated directly from the means and the covariances of the data series.

To derive the unconditional moments of the jump-diffusion returns processes in (4), we identify the characteristic function by exploiting its relation to the Kolmogorov backward equation.[14] Differentiating the characteristic function then gives the moments of the returns process. The expressions for the moments of the continuously compounded returns are the following: for $n, m = \{1, \ldots, N\}$,

$$\text{mean} = t\left(\alpha_n - \tfrac{1}{2}\sigma_n^2 + \lambda\mu_n\right), \tag{22}$$

$$\text{covariance} = t\,[\sigma_{nm} + \lambda(\mu_n\mu_m + \nu_n\nu_m)], \tag{23}$$

[13] The negative skewness arises also because volatility tends to be higher when returns are negative.

[14] Details of this derivation are given in Das and Uppal (2003).

Table II

Descriptive Statistics for Equity Returns—Multivariate

Panel A gives the covariances and the correlations (in italics) between U.S. dollar returns for the developed-country indexes and Panel B gives this for the emerging-country indexes. The data for the developed countries are for the period January 1982 to February 1997, and include 182 observations of month-end values of the equity indexes for the United States (U.S.), United Kingdom (U.K.), Japan (JP), Germany (GE), Switzerland (SW), and France (FR). The data for emerging economies consist of 227 observations of the beginning-of-month value of the equity indexes for the USA, Argentina (ARG), Hong Kong (HKG), Mexico (MEX), Singapore (SNG), and Thailand (THA) for the period January 1980 to December 1998.

Panel A: Developed Countries—Covariances (normal) and Correlations (italics)							
	U.S.	U.K.	JP	GE	SW	FR	Avg. Correl.
U.S.	0.0018	*0.5750*	*0.3348*	*0.4274*	*0.5472*	*0.5173*	
U.K.	0.0013	0.0032	*0.4468*	*0.4916*	*0.5593*	*0.5507*	
JP	0.0009	0.0017	0.0049	*0.3815*	*0.4442*	*0.4765*	
GE	0.0010	0.0016	0.0015	0.0033	*0.6873*	*0.6593*	
SW	0.0011	0.0016	0.0015	0.0020	0.0027	*0.6161*	*0.5142*
FR	0.0013	0.0019	0.0020	0.0023	0.0019	0.0037	
Panel B: Emerging Countries—Covariances (normal) and Correlations (italics)							
	USA	ARG	HKG	MEX	SNG	THA	Avg. Correl.
USA	0.0017	*0.1039*	*0.4051*	*0.3586*	*0.5519*	*0.3445*	
ARG	0.0009	0.0464	*0.0580*	*0.2167*	*0.0842*	*0.1286*	
HKG	0.0017	0.0012	0.0105	*0.2475*	*0.5479*	*0.4347*	
MEX	0.0021	0.0067	0.0036	0.0206	*0.3543*	*0.2972*	
SNG	0.0017	0.0014	0.0043	0.0039	0.0060	*0.5291*	*0.3109*
THA	0.0014	0.0028	0.0046	0.0044	0.0042	0.0108	

$$\text{coskewness} = \frac{t\lambda\left[2\mu_n\nu_n\nu_m + \mu_m\left(\mu_n^2 + \nu_n^2\right)\right]}{\text{variance}_n\text{variance}_m^{1/2}}, \tag{24}$$

$$\text{excess kurtosis} = \frac{t\lambda\left(3\nu_n^4 + 6\nu_n^2\mu_n^2 + \mu_n^4\right)}{\text{variance}_n^2}. \tag{25}$$

Comparing the mean and covariance for the jump-diffusion processes considered above with those for the pure-diffusion processes ($\lambda = 0$), one gets the following:

$$\alpha_n^{\mathsf{J}} = \lambda\mu_n, \tag{26}$$

$$\sigma_{nm}^{\mathsf{J}} = \lambda\left(\mu_n\mu_m + \nu_n\nu_m\right). \tag{27}$$

With this compensation in equations (7) and (8), the expected returns, variances, and covariances will be the same under the jump-diffusion and pure-diffusion processes.

For the jump-diffusion process, the parameters to be estimated are $\{\lambda, \alpha, \Sigma, \mu, \nu\}$, from the moment conditions in equations (22) to (25). In our experiment

Table III

Parameter Estimates for the Returns Processes

This table reports estimates of the parameters for the multivariate system of jump-diffusion asset returns, $\{\lambda, \mu, \nu\}$ obtained by minimizing the square of the difference between the moment conditions in equations (24) and (25) and the moments implied by the data. Panel A gives the estimates for developed-country return indexes and Panel B gives the estimates for emerging economies. In addition to the parameter estimates, the table reports the reconstructed moments that are obtained by substituting the parameters estimated into equations (24) and (25), which are then compared to the moments of the data.

Panel A: Developed Countries							
	U.S.	U.K.	JP	GE	SW	FR	Avg.
λ	0.0501						0.0501
μ	−0.0660	−0.0797	0.0043	−0.0344	−0.0466	−0.0675	−0.0483
ν	0.0914	0.0792	0.1075	0.1167	0.1185	0.0902	0.1006
Skewness: reconstructed	−1.3160	−0.5496	0.0222	−0.3782	−0.7567	−0.4291	−0.5679
Skewness: in data	−1.1648	−0.4623	−0.0508	−0.2308	−0.6382	−0.4325	−0.4966
Excess kurtosis: reconstructed	7.2148	1.9182	0.8540	2.9662	5.5474	1.5872	3.3480
Excess kurtosis: in data	7.2236	1.9212	0.8754	2.9546	5.5405	1.5780	3.3489

Panel B: Emerging Countries							
	USA	ARG	HKG	MEX	SNG	THA	Avg.
λ	0.0138						0.0138
μ	−0.1280	0.2292	−0.2295	−0.2631	−0.1576	−0.1107	−0.1099
ν	0.0919	0.7179	0.3001	0.4929	0.2068	0.3009	0.3518
Skewness: reconstructed	−1.0434	0.5085	−0.9507	−0.9806	−0.7272	−0.3903	−0.5973
Skewness: in data	−1.1353	0.1187	−1.4163	−2.0224	−0.7684	−0.6077	−0.9719
Excess kurtosis: reconstructed	6.1980	6.2009	6.9511	9.1905	4.8729	3.7623	6.1960
Excess kurtosis: in data	6.1823	6.2377	6.9388	9.1851	4.8603	3.7800	6.1974

we wish to set equal the means and covariances of the jump-diffusion processes to those from the pure-diffusion process; hence set $\alpha = \hat{\alpha} - \lambda\mu$ and $\boldsymbol{\Sigma} = \widehat{\boldsymbol{\Sigma}} - \lambda(\mu\mu' + \nu\nu')$. Thus, we need to estimate only λ, and the 6×1 vectors μ and ν (a total of 13 parameters) to match the 6×1 kurtosis and 6×6 coskewness conditions for a total of 42 moment conditions.[15] We choose these 13 parameters to minimize the squared deviation of the 42 moment conditions from their values implied by the data. Once we have these parameters, we can obtain α by subtracting $\lambda\mu$ from $\hat{\alpha}$, and $\boldsymbol{\Sigma}$ by subtracting $\lambda(\mu\mu' + \nu\nu')$ from $\widehat{\boldsymbol{\Sigma}}$.

Table III reports the parameter estimates obtained using the method of moments. From Panel A of this table we see that for the developed countries, the estimated value of $\lambda = 0.0501$, and this is significantly different from 0. The estimated value for λ of 0.0501 indicates that on average the chance of a jump in any month is about 5%, or one jump is expected every 20 months. Our estimate of λ is lower than that in studies estimating the likelihood of a jump in

[15] Note that coskewness between asset n and m is different from that between m and n; thus, the coskewness matrix contains 6 skewness terms on the diagonal, and 30 unique coskewness terms.

the returns series of a single index, which is typically on the order of 0.20; the reason is that in our model λ measures the likelihood of a *systemic* jump rather than an idiosyncratic jump. The average expected jump size across countries is -0.0483, while the volatility of the jump size is 0.1006.

From Panel B of Table III, we see that the estimated value of λ in the data for emerging markets is 0.0138, lower than that for developed countries. This is not surprising given that linkages between emerging countries are much weaker than those for the developed countries considered in our sample. Observe, however, from the comparison of the parameter estimates in Panel A for the developed countries to those for emerging markets in Panel B that the average jump size (μ) is more than double for emerging markets—which reflects the higher skewness in this data set. Also, the average volatility of jumps (ν) is three times higher for emerging markets, and this reflects the higher kurtosis in this data. Thus, even though systemic jumps are less frequent for emerging countries than for developed countries, their average (absolute) sizes and volatilities are much larger. Comparing the relative size of the covariation coming from systemic jumps, σ^{J}_{nm}, to overall covariation, $\hat{\sigma}_{nm}$, we find that on average this is 0.37 for developed markets and 0.55 for emerging markets, indicating that systemic jumps contribute more to the overall covariation of returns in emerging markets than in developed markets.

To measure how well the estimated parameters do at matching the moments of the data, we use the estimated parameters and the moment conditions in (24) and (25) to reconstruct the skewness and kurtosis measures.[16] Comparing these reconstructed moments with the estimated moments, we see that the model does quite well in matching the kurtosis in the data but is less successful in matching the skewness. Studying the averages reported in the last column of the table, we observe that the kurtosis is matched almost exactly for both developed and emerging countries, while the magnitude of skewness from the model is greater than that in the data for the developed countries (Panel A), and smaller for emerging countries (Panel B). Because the moments are not matched exactly, we will evaluate in Section III.C the sensitivity of our results to these parameter estimates.

B. Portfolio Weights and Certainty-Equivalent Cost

In our calibration exercise, the parameters we use for the returns process are those reported in Table III. In addition to this, we need to specify the risk-free rate and the agent's relative risk aversion. We assume that the monthly riskless interest rate for the U.S. investor is $0.06/12 = 0.005$, which is close to the average U.S. 1-month riskless interest rate in our data, and we set the base-case relative risk aversion, γ, equal to 3.0.

[16] The means and covariances are matched exactly by construction and so are not reported. The results on the comparison of coskewness from the model to that in the data are not reported but are similar to those for skewness.

With these parameter values, we solve numerically the first-order conditions in Proposition 2 to obtain the optimal portfolio weights, $\mathbf{w}$, for an investor who accounts for systemic risk.[17] We also compute $\hat{\mathbf{w}}$, the weights of the investor who ignores systemic jumps and assumes returns are given by a pure-diffusion process. In addition to these portfolio weights, we also report the composition of the portfolio consisting of *only* risky assets, which can be obtained by dividing each individual weight by the total investment in risky assets. These weights are given by $\mathbf{w}/(\mathbf{w}'\mathbf{1})$ for the systemic-jump case and $\hat{\mathbf{w}}/(\hat{\mathbf{w}}'\mathbf{1})$ for the pure-diffusion case.

Table IV reports the weights for developed-country indexes in Panel A and those for the emerging countries in Panel B. Within each panel, the portfolio weights are given for three different levels of risk aversion: $\gamma = \{1, 3, 5\}$. To evaluate the effect of systemic risk on diversification, we compare the quantities $\mathbf{w}/(\mathbf{w}'\mathbf{1})$ with $\hat{\mathbf{w}}/(\hat{\mathbf{w}}'\mathbf{1})$, which are reported in the last two rows for each set of results. We observe that the effect of systemic jumps on the composition of the risky-asset portfolio is not substantial: For instance, in Panel A of Table IV for the case with risk aversion of 3, the change in all the portfolio weights is less than 0.03. This is also true for more risk-averse investors; only in the case of the investor with risk aversion of unity is there some change in the composition of the risky-asset portfolio, but even in this case the allocation to the U.S. home asset increases by only 0.023, from 0.770 to 0.793. In the case of emerging markets in Panel B of Table IV, the effect of systemic risk is bigger: For the investor with a risk aversion of 3, the portfolio weights change by up to 0.07; for instance, the investment in the U.S. index drops from 1.452 to 1.379 when an investor accounts for systemic risk. The reason for the bigger change in these portfolio weights is that jumps are much larger in emerging countries compared to those in developed countries and contribute much more to the overall covariation in returns.

Table IV also gives the investment in the riskless asset and the total invested across all risky assets. Observe that for developed countries, the total investment in risky assets (last column) is smaller for the investor who accounts for systemic risk. For example, in Panel A, when relative risk aversion equals 3, the investment in risky assets is 1.141 for the investor who ignores systemic risk as opposed to only 1.104 for the investor who accounts for this. Moreover, this effect is much more pronounced for the aggressive investor with unit risk aversion who holds a levered portfolio (3.423 versus 2.814) than it is for the conservative investor with a risk aversion of 5 (0.685 versus 0.678). Thus, as one would expect, the investor who recognizes the presence of systemic risk reduces leverage in order to ensure that wealth is positive in the event of an adverse systemic shock. Of course, because the diffusion investor ignores systemic risk, her portfolio has too much leverage, and in the event of a large

[17] In our numerical work, we use a discretization of the normal distribution with the discrete points ranging over plus/minus three standard deviations. Because of the finite support assumed in this approximation, even the portfolio of the investor who accounts for systemic risk can have levered positions without these positions leading to negative wealth. Of course, ignoring systemic jumps altogether can lead to negative wealth.

Table IV

Portfolio Weights

This table gives the portfolio weights for an investor who chooses investments in six equity indexes and the riskless asset to maximize expected utility of terminal wealth. The weights are reported for two data sets: in Panel A, for a portfolio diversified across equity indexes of developed countries; in Panel B, for a portfolio diversified across indexes for emerging countries. In each panel, the first two rows of weights give (1) the optimal weights, **w**, for an investor who accounts for systemic jumps; and (b) the weights $\hat{\mathbf{w}}$, for an investor who ignores systemic jumps and assumes a pure-diffusion process for returns. For these two sets of weights, the next two rows of the table give the composition of the risky-asset portfolio, which is obtained by dividing the weight for each index by the total investment in risky assets. The riskless interest rate is assumed to be 0.005 per month.

Panel A: Developed Countries

Country	U.S.	U.K.	JP	GE	SW	FR	Riskless	Total in Risky Assets
	Relative Risk Aversion = 1							
Systemic weights: **w**	2.231	−1.012	−0.212	1.556	−0.102	0.353	−1.814	2.814
Diffusion weights: $\hat{\mathbf{w}}$	2.635	−0.977	−0.218	1.625	0.008	0.348	−2.423	3.423
Risky-asset portfolio: $\frac{\mathbf{w}}{\mathbf{w}'\mathbf{1}}$	0.793	−0.360	−0.075	0.553	−0.036	0.125	0	1
Risky-asset portfolio: $\frac{\hat{\mathbf{w}}}{\hat{\mathbf{w}}'\mathbf{1}}$	0.770	−0.285	−0.064	0.475	0.002	0.102	0	1
	Relative Risk Aversion = 3							
Systemic weights: **w**	0.847	−0.336	−0.068	0.543	0.011	0.107	−0.104	1.104
Diffusion weights: $\hat{\mathbf{w}}$	0.878	−0.326	−0.073	0.542	0.003	0.116	−0.141	1.141
Risky-asset portfolio: $\frac{\mathbf{w}}{\mathbf{w}'\mathbf{1}}$	0.767	−0.304	−0.061	0.492	0.010	0.097	0	1
Risky-asset portfolio: $\frac{\hat{\mathbf{w}}}{\hat{\mathbf{w}}'\mathbf{1}}$	0.770	−0.285	−0.064	0.475	0.002	0.102	0	1
	Relative Risk Aversion = 5							
Systemic weights: **w**	0.518	−0.201	−0.041	0.328	0.010	0.064	0.322	0.678
Diffusion weights: $\hat{\mathbf{w}}$	0.527	−0.195	−0.044	0.325	0.002	0.070	0.315	0.685
Risky-asset portfolio: $\frac{\mathbf{w}}{\mathbf{w}'\mathbf{1}}$	0.764	−0.297	−0.060	0.483	0.015	0.094	0	1
Risky-asset portfolio: $\frac{\hat{\mathbf{w}}}{\hat{\mathbf{w}}'\mathbf{1}}$	0.770	−0.285	−0.064	0.475	0.002	0.102	0	1

Panel B: Emerging Countries

Country	USA	ARG	HKG	MEX	SNG	THA	Riskless	Total in Risky Assets
	Relative Risk Aversion = 1							
Systemic weights: **w**	4.294	0.120	0.209	−0.267	−0.406	−0.893	−2.056	3.056
Diffusion weights: $\hat{\mathbf{w}}$	4.563	−0.006	0.156	−0.314	−0.380	−0.876	−2.143	3.143
Risky-asset portfolio: $\frac{\mathbf{w}}{\mathbf{w}'\mathbf{1}}$	1.405	0.039	0.068	−0.088	−0.133	−0.292	0	1
Risky-asset portfolio: $\frac{\hat{\mathbf{w}}}{\hat{\mathbf{w}}'\mathbf{1}}$	1.452	−0.002	0.050	−0.100	−0.121	−0.279	0	1
	Relative Risk Aversion = 3							
Systemic: **w**	1.457	0.039	0.076	−0.085	−0.134	−0.296	−0.056	1.056
Diffusion: $\hat{\mathbf{w}}$	1.521	−0.002	0.052	−0.105	−0.127	−0.292	−0.048	1.048
Risky-asset portfolio: $\frac{\mathbf{w}}{\mathbf{w}'\mathbf{1}}$	1.379	0.037	0.072	−0.080	−0.127	−0.281	0	1
Risky-asset portfolio: $\frac{\hat{\mathbf{w}}}{\hat{\mathbf{w}}'\mathbf{1}}$	1.452	−0.002	0.050	−0.100	−0.121	−0.279	0	1
	Relative Risk Aversion = 5							
Systemic weights: **w**	0.877	0.024	0.046	−0.051	−0.081	−0.178	0.362	0.638
Diffusion weights: $\hat{\mathbf{w}}$	0.913	−0.001	0.031	−0.063	−0.076	−0.175	0.371	0.629
Risky-asset portfolio: $\frac{\mathbf{w}}{\mathbf{w}'\mathbf{1}}$	1.375	0.037	0.072	−0.079	−0.126	−0.279	0	1
Risky-asset portfolio: $\frac{\hat{\mathbf{w}}}{\hat{\mathbf{w}}'\mathbf{1}}$	1.452	−0.002	0.050	−0.100	−0.121	−0.279	0	1

negative systemic jump she may lose all her wealth. In the case of emerging markets, reported in Panel B of Table IV, the effect on leverage is much weaker: While leverage decreases in the case of the investor with a risk aversion of unity, there is almost no change for the more conservative investors who have a risk aversion of 3 and 5.

To evaluate the magnitude of the effect on lifetime utility of the portfolio strategy that accounts for systemic risk relative to the strategy that ignores this effect, we compute the quantity CEQ (defined in equation (20)), which measures the additional wealth required at $t = 0$ to raise the lifetime utility of the investor following the suboptimal portfolio strategy, $\hat{\mathbf{w}}$, to the level of the investor choosing the portfolio $\mathbf{w}$, which accounts for systemic risk. This quantity depends on the time horizon of the investor and we report it in Table V for horizons of 1 to 5 years and for relative risk aversion also ranging from 1 to 5.

From Panel A of Table V, we see that in the case of developed economies, and for the base-case relative risk aversion of 3, CEQ is equal to 0.00007 for a horizon of 1 year, and increases to 0.00033 for a horizon of 5 years. That is, for an investor with an initial wealth of $1,000, the cost of ignoring systemic risk is $0.07 if the horizon is 1 year and $0.33 if the horizon is 5 years. In the case of emerging economies, reported in Panel B of Table V, the CEQ is bigger: For an initial wealth of $1,000 and relative risk aversion of 3, the cost

Table V

Certainty Equivalent (CEQ) Cost of Ignoring Systemic Risk

This table gives the CEQ for an investor with an initial wealth of $1 who chooses investments in six equity indexes and the riskless asset to maximize expected utility of terminal wealth. The CEQ measures the additional initial wealth, per dollar of investment, in order to raise the utility of an investor who ignores systemic risk to the level of an investor who recognizes this risk. The table reports the CEQ for investment horizons of 1 to 5 years and for levels of relative risk aversion (γ) from 1 to 5. The CEQ are reported for two cases: in Panel A, for a portfolio diversified across equity indexes of developed countries; in Panel B, for a portfolio diversified across indexes for emerging countries. The riskless interest rate is assumed to be 0.005 per month. The asterisks denote that the portfolio weights chosen by the investor who ignored systemic risk led to negative wealth.

	Investment Horizon				
RRA (γ)	1 year	2 years	3 years	4 years	5 years
			Panel A: Developed Countries		
1	*	*	*	*	*
2	0.00034	0.00069	0.00103	0.00138	0.00172
3	0.00007	0.00013	0.00020	0.00026	0.00033
4	0.00002	0.00005	0.00007	0.00009	0.00011
5	0.00001	0.00002	0.00003	0.00005	0.00006
			Panel B: Emerging Countries		
1	*	*	*	*	*
2	0.00498	0.00998	0.01501	0.02006	0.02514
3	0.00316	0.00633	0.00951	0.01269	0.01589
4	0.00232	0.00465	0.00698	0.00932	0.01166
5	0.00184	0.00368	0.00552	0.00737	0.00922

of ignoring systemic risk ranges from \$3.16 for a horizon of 1 year to \$15.89 for a 5-year horizon. The cost of ignoring systemic risk is larger in the case of emerging countries because the mean jump size and its variance are much larger in these markets.

For both data sets, the CEQ decreases as risk aversion increases. The intuition is that as risk aversion increases, the investor holds a smaller proportion of her wealth in risky assets; hence, the exposure to systemic risk, and its effect on CEQ, is smaller. On the other hand, as explained in Liu et al. (2003), when risk aversion is low, the investor would like to hold a levered position in risky assets, and an investor who ignores systemic risk may lever her portfolio to such an extent that an adverse systemic shock leads to ruin. The asterisk in the first row of Panels A and B of Table V indicates that in the case where risk aversion equals unity, the investor who ignores systemic risk chooses a portfolio with so much leverage that for large negative systemic jumps, her wealth drops below zero. Thus, for investors with low risk aversion, the cost of ignoring systemic risk can be substantial.

C. Verifying the Robustness of the Results

In this subsection, we examine whether the results reported above are sensitive to (1) the choice of data; (2) the assumption about unrestricted borrowing and short selling; (3) the estimates of expected returns ($\hat{\alpha}$), which are notoriously difficult to estimate precisely; and (4) the estimates for $\{\lambda, \mu, \nu\}$, the parameters driving the systemic jump. The main conclusion of this robustness exercise is that none of these factors materially change the conclusions drawn from Table IV or the CEQ cost implied by these portfolio weights reported in Table V. We discuss each robustness check below and the details are provided in Das and Uppal (2003).

In our analysis, for robustness we have considered two data sets: one for developed countries and the other for emerging countries. Comparing the parameter estimates for the returns process for these two sets of data, we see that *systemic* jumps are more likely across developed markets than for the emerging markets considered in our sample, though the expected jump size and volatility are larger for emerging countries. Our results on portfolio choice are broadly similar across these two data sets: the effect of systemic risk on portfolio weights is not large for risk aversion greater than 2, and thus, the initial wealth required to compensate for systemic risk is also small, especially in the case of emerging countries.

The portfolios we have examined in Table IV were unconstrained, but in practice an investor may face constraints on borrowing or short selling risky assets. When we compute the optimal portfolio in the presence of constraints on borrowing and short sales, we find that, as in the unconstrained case, there is only a small difference between the optimal portfolio $\mathbf{w}$ that accounts for systemic jumps, and $\hat{\mathbf{w}}$, the portfolio where this risk is ignored. For both developed and emerging countries, the difference is smaller in the presence of constraints than it was for portfolios that were unconstrained.

Thus, with constraints on short sales and borrowing, the result of which is to reduce levered portfolio positions, the effect of systemic jumps on portfolio weights and CEQ is even smaller.

Another concern is that the portfolio weights computed in Table IV could potentially be sensitive to the estimate of the expected return, leading to large imbalances in the weights assigned to risky assets. There is a large literature discussing the problems in estimating expected returns, the extreme portfolios generated by this, and ways to reduce it.[18] To examine the sensitivity of our results to the estimates of expected returns, we recompute the portfolio weights by *averaging* the estimates of expected returns across all assets and using this average as a proxy for the expected return on all the assets. Thus, all assets have the same expected return, so that the weights are no longer driven by differences in expected returns. Again, because reducing differences in expected returns reduces extreme portfolio positions, the CEQ numbers are even smaller than those reported in Table V.

Above, we have examined the robustness of our results to the estimates of expected returns. We now explore the effect of the estimates for the three new parameters introduced by our jump-diffusion model: λ, which dictates the frequency of jumps, μ, which measures the expected size of jumps, and ν, which measures the volatility of the jump size. In this analysis, relative risk aversion is assumed to be 3 and the risk-free rate is equal to 0.005 per month, which matches the assumptions for the base case considered in Table IV.

In order to make it possible to use graphs so that we can report the weights for a wide range of parameter values, we consider a situation in which an investor has to allocate wealth to one risk-free asset and only two risky assets. Moreover, the two risky assets are assumed to be symmetric, with parameter values for their returns process being the average of the values estimated for the six developed-country indexes given in Panel A of Table III. Using these averages as the base case, in our experiment we evaluate in Figures 1 to 3, the portfolio weights and CEQ cost for a range of values for λ, μ, and ν. Each of these figures has two panels: in the first, we report the optimal portfolio weights of an investor who accounts for systemic jumps (solid line in figure) and an investor who ignores this (flat dotted line in figure), and in the second panel we plot the CEQ corresponding to these portfolio weights for investors with a horizon of 1, 2, and 5 years.

In Figure 1, we vary the jump-intensity parameter from 0 to 0.25, which is five times its estimated value of 0.05. To gauge whether this range is broad enough, note that a λ of 0.25 implies that there are about three systemic jumps each year, and corresponds to skewness and kurtosis that are five times their value in the data. The figure shows that as λ increases, the difference between the two portfolio strategies increases. For $\lambda = 0.10$, which is double its estimated value, the difference in portfolio weights is 0.03 and the CEQ for an initial investment

[18] An early reference to this problem is given in Jorion (1985) and Dumas and Jacquillat (1990). Green and Hollifield (1992) provide a good discussion of this problem, and Connor (1997) proposes a nice solution to it.

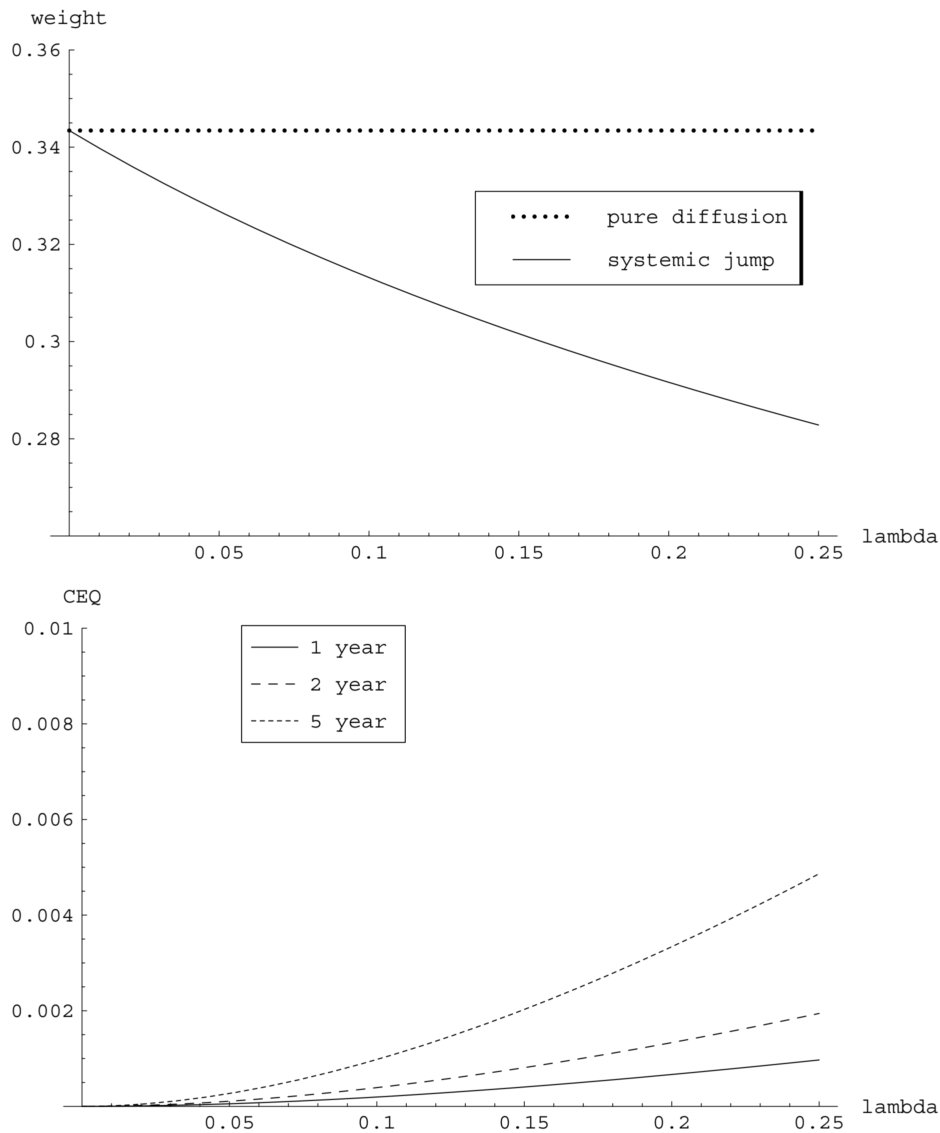

Figure 1. Portfolio weights and CEQ with respect to jump intensity. The top plot of the figure gives the portfolio of an investor who accounts for systemic jumps (solid line) and an investor who ignores this and models returns as a pure diffusion (dotted line). The case considered is one in which there are only *two* risky assets. The parameters for the returns processes for both assets are calibrated to the *average* of the estimates for developed countries, reported in the last column of Panel A of Table III, with the exception of λ, which is allowed to range from 0 to 0.25, corresponding to skewness and kurtosis ranging from 0 to 5 times their estimates in the data. The lower plot shows the corresponding CEQ for these two portfolios, where CEQ measures the percentage of initial wealth that needs to be given to the investor who ignores systemic risk to make him as well off as the investor who accounts for systemic risk. We assume that relative risk aversion, γ, is 3 and that the riskless interest rate is 0.005 per month.

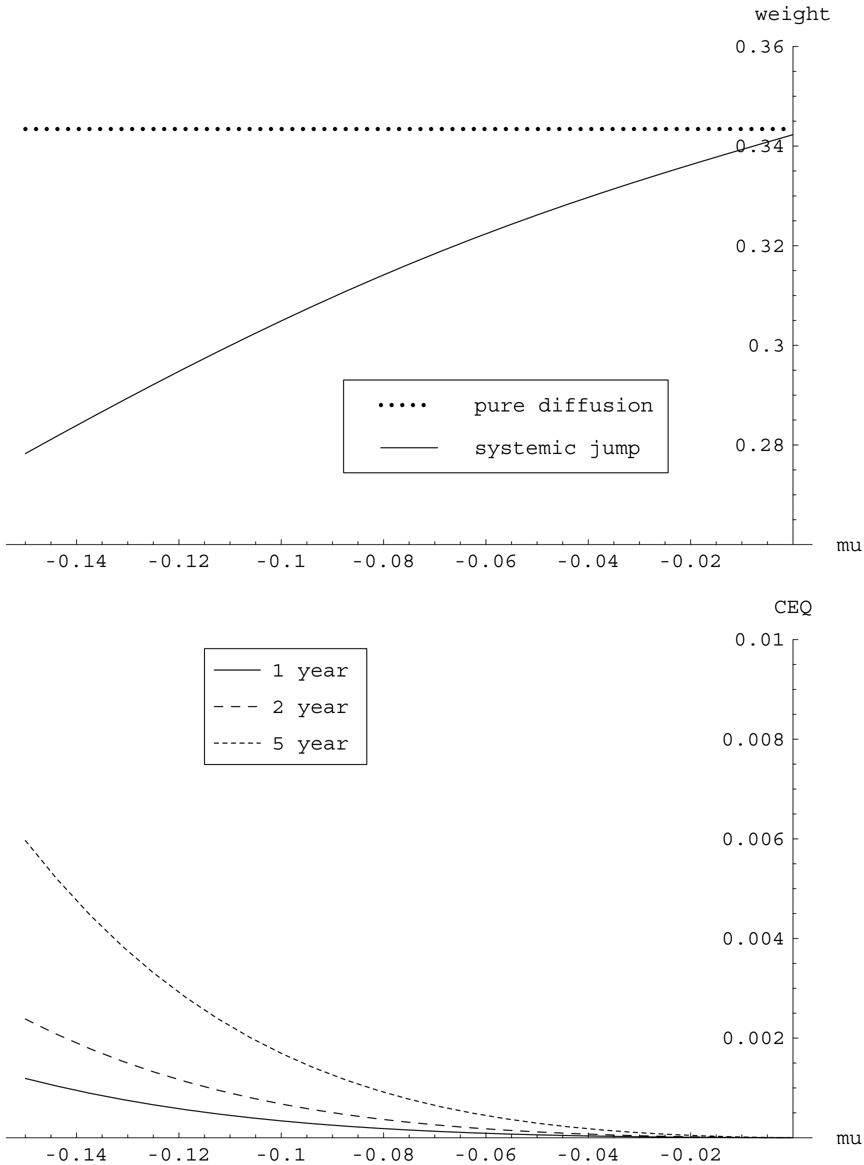

Figure 2. Portfolio weights and CEQ with respect to mean of jump size. The top plot of the figure gives the portfolio of an investor who accounts for systemic jumps (solid line) and an investor who ignores this and models returns as a pure diffusion (dotted line). The case considered is one in which there are only *two* risky assets. The parameters for the returns processes on both assets are calibrated to the *average* of the estimates for developed countries, reported in the last column of Panel A of Table III, with the exception of μ, which is allowed to range from 0 to -0.15, corresponding to skewness ranging from 0 to 4.8 times its estimate in the data, and kurtosis ranging from 0 to 7.1 times its estimated value. The lower plot shows the corresponding CEQ for these two portfolios, where CEQ measures the percentage of initial wealth that needs to be given to the investor who ignores systemic risk to make him as well off as the investor who accounts for systemic risk. We assume that relative risk aversion, γ, is 3 and that the riskless interest rate is 0.005 per month.

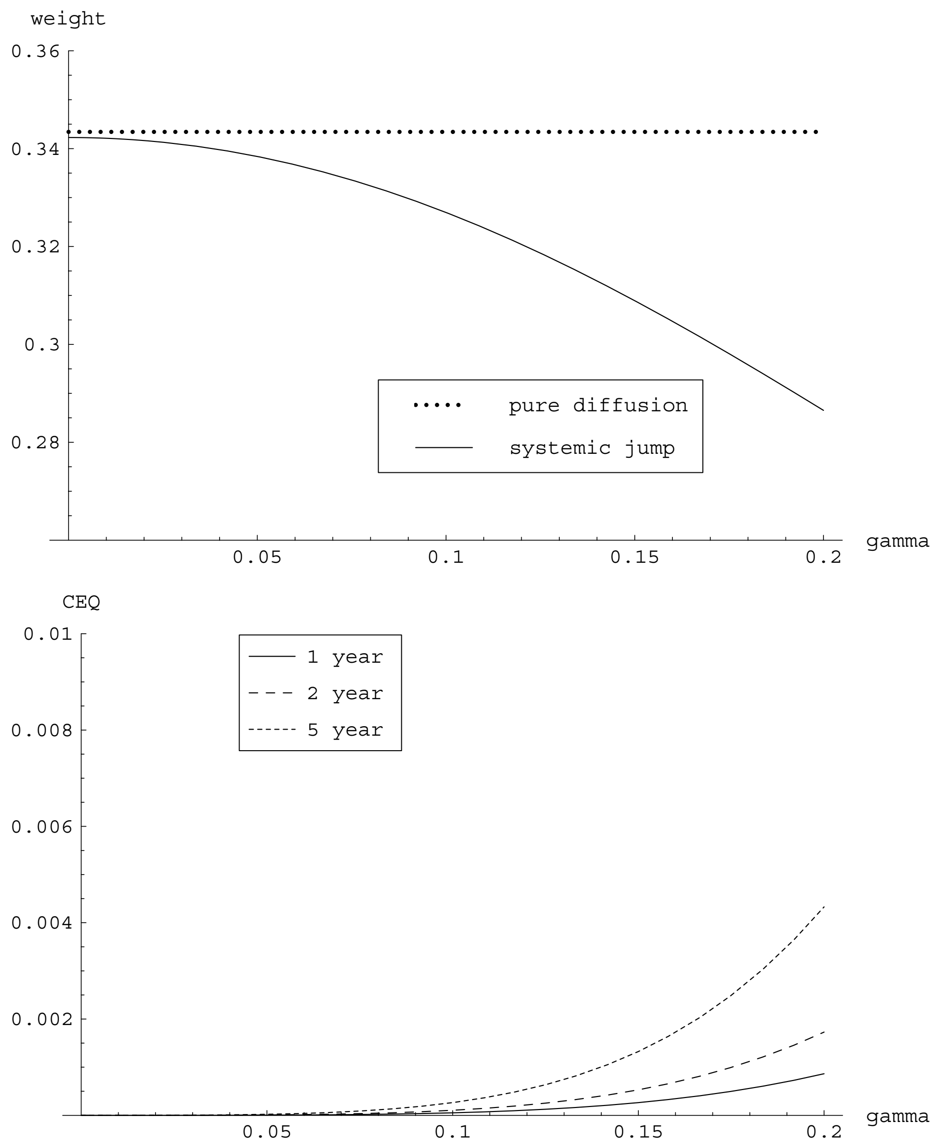

Figure 3. Portfolio weights and CEQ with respect to variance of jump size. The top plot of the figure gives the portfolio of an investor who accounts for systemic jumps (solid line) and an investor who ignores this and models returns as a pure diffusion (dotted line). The case considered is one in which there are only *two* risky assets. The parameters for the returns processes for both assets are calibrated to the *average* of the estimates for developed countries, reported in the last column of Panel A of Table III, with the exception of ν, which is allowed to range from 0 to 0.20, corresponding to skewness ranging from 0 to 3.8 times its estimate in the data, and kurtosis ranging from 0 to 11.8 times its estimated value. The lower plot shows the corresponding CEQ for these two portfolios, where CEQ measures the percentage of initial wealth that needs to be given to the investor who ignores systemic risk to make him as well off as the investor who accounts for systemic risk. We assume that relative risk aversion, γ, is 3 and that the riskless interest rate is 0.005 per month.

of \$1,000 is only \$1 for an investor with a horizon of 5 years, and is even smaller for shorter horizons. For the extreme value of $\lambda = 0.25$, the difference in the two portfolio weights is 0.06; that is, a pure-diffusion investor would invest 0.34 in each of the risky assets and 0.32 ($=1 - 2 \times 0.34$) in the risk-free asset, while an investor who accounts for systemic risk would invest only 0.28 in the risky assets and 0.44 ($=1 - 2 \times 0.28$) in the risk-free asset. The CEQ in this extreme case for an investor with a horizon of 1 year is \$1, and the CEQ for a horizon of 5 years is \$5. Thus, we conclude that small deviations from the estimated value of λ will not have a large effect on our conclusions about portfolio weights and the corresponding CEQ cost.

Figure 2 considers the effect of μ, the parameter for the expected jump size that determines the sign for the skewness of returns. The average of this parameter in the data for developed countries is about -0.05 (Panel A of Table III) and we allow this to vary from 0 to -0.15, three times its estimated value. The value of $\mu = -0.15$ implies that skewness is 4.8 times its estimated magnitude and that kurtosis is 7.1 times what it is in the data. For $\mu = -0.10$, the difference in portfolio weights is about 0.03, and for $\mu = -0.15$, we find that the difference in the portfolio weights is 0.06. The effect on the CEQ is more sensitive to μ than it was to λ: for the extreme case where $\mu = -0.15$ and the investor has a horizon of 5 years, CEQ = \$6 for an initial investment of \$1,000; for a more reasonable level of $\mu = -0.10$, CEQ = \$1.80 for an investor with a 5-year horizon; and is only \$0.40 for an investor with a 1-year horizon.

Finally, in Figure 3 we vary the volatility of the jump size, ν, from 0 to 0.20, which is twice its estimated value and corresponds to 3.8 times the estimated value of skewness and 11.8 times the estimate of kurtosis. The effect of this parameter on the portfolio weights and CEQ cost is smaller than that of λ and μ. For $\nu = 0.20$, the difference in the portfolio weights of systemic and pure-diffusion investors is about 0.05, and the CEQ is \$5 for an investor with a 5-year horizon and \$0.90 for an investor with a 1-year horizon.

Based on the above exercise, we conclude that our findings about the effect of systemic risk on the optimal portfolio composition and on CEQ are robust to reasonable deviations from the estimated values of the parameters.

IV. Conclusion

Returns on international equities are characterized by jumps occurring at the same time across countries, leading to return distributions that are fat-tailed and negatively skewed. We develop a simple and parsimonious model of asset returns to capture these empirical properties, and then show how an investor would choose an optimal portfolio when returns have these features. We apply the proposed method to determine the weights for a portfolio allocated over a riskless asset, an equity index for the U.S. and five international equity indexes. We consider two sets of international indexes, the first for developed countries and the second for emerging countries, and calibrate these data sets to two models: one that incorporates systemic risk and the other that ignores it.

The main result from our analysis is that incorporating systemic risk has two effects: (1) it reduces the gains from diversifying across a range of assets, and (2) it makes leveraged portfolios much more susceptible to large losses. Upon calibrating our model to index returns for developed economies and also for emerging economies, we find that the loss from the reduction in diversification is not substantial. However, for investors with low risk aversion who desire levered positions, the cost of ignoring systemic risk is much larger, and in the case of a highly levered portfolio, there is a positive probability of losing one's entire wealth if there is a large negative systemic shock.

Appendix

Proof of Proposition 1: Equating the expressions in (5) and (6) to those for the pure-diffusion returns process in equations (2) and (3) gives the result. Q.E.D.

Proof of Proposition 2: Simplifying the jump term in the Bellman equation (13), using the conjecture that the value function is of the form $V(W_t, t) = A(t)\frac{W_t^{1-\gamma}}{1-\gamma}$, we get

$$\begin{aligned}\lambda E[V(W_t + W_t \mathbf{w}'\mathbf{J}_t, t) - V(W_t, t)] &= \lambda E[V(W_t[1+\mathbf{w}'\mathbf{J}_t], t) - V(W_t, t)] \\ &= \lambda \frac{A(t)W_t^{1-\gamma}}{1-\gamma} E\big[(1+\mathbf{w}'\mathbf{J}_t)^{1-\gamma} - 1\big] \\ &= \lambda V(W_t, t) E\big[(1+\mathbf{w}'\mathbf{J}_t)^{1-\gamma} - 1\big]. \end{aligned} \tag{A1}$$

After substituting (A1) into (13), one obtains

$$\begin{aligned} 0 = \max_{\{\mathbf{w}\}} \Bigg\{ &\frac{\partial V(W_t, t)}{\partial t} + \frac{\partial V(W_t, t)}{\partial W} W_t[\mathbf{w}'\mathbf{R} + r] + \frac{1}{2}\frac{\partial^2 V(W_t, t)}{\partial W^2} W_t^2 \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} \\ &+ \lambda V(W_t, t) E\big[(1+\mathbf{w}'\mathbf{J}_t)^{1-\gamma} - 1\big]\Bigg\}. \end{aligned} \tag{A2}$$

Substituting the functional form of the value function into (A2) gives equation (15). Differentiating this equation gives the result in the proposition.

To identify $A(t)$, we start by evaluating (15) at the optimal portfolio weights, $\mathbf{w}$, which implies:

$$\frac{1}{A(t)}\frac{dA(t)}{dt} = -\kappa, \tag{A3}$$

where

$$\kappa \equiv (1-\gamma)[\mathbf{w}'\mathbf{R} + r] - \tfrac{1}{2}\gamma(1-\gamma)\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} + \lambda E\big[(1+\mathbf{w}'\mathbf{J}_t)^{1-\gamma} - 1\big]. \tag{A4}$$

Integrating then gives

$$A(t) = ae^{-\kappa t}, \tag{A5}$$

where a is the constant of integration. Using the boundary condition that

$$A(T) = ae^{-\kappa T} = 1 \tag{A6}$$

then implies that

$$a = e^{\kappa T} \tag{A7}$$

so that

$$A(t) = e^{\kappa(T-t)} \tag{A8}$$

and the value function is

$$V(W_t, t) = e^{\kappa(T-t)} \frac{W_t^{1-\gamma}}{1-\gamma} \tag{A9}$$

with κ defined in equation (A4). Q.E.D.

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