Credit Risk Derivatives

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This article models the pricing of derivatives on credit risk, instruments proposed in 1992 by the International Swap Dealers Association that have started attracting market attention. The exact structure of the instruments continues to evolve today.

We develop a framework to understand the key features of this class of products. It is shown that the price of the credit risk option is the expected forward value of a put option on a risky bond with a credit level-adjusted exercise price. Stripping of credit risk from the total risk of the bond is enabled by employing a stochastic strike price for the credit risk option. The article provides a framework for trading and hedging credit risk.

The pricing of corporate default risk is receiving increasing attention in the finance literature. A number of articles have attempted varied approaches to pricing risky debt (see, for example, Jarrow, Lando, and Turnbull [1994], Jarrow and Turnbull [1994], Kim, Ramaswamy, and Sundaresan [1993], Longstaff and Schwartz [1993], Nielsen, Saa-Requejo, and Santa-Clara [1993], and Shimko, Tejima, and van Deventer [1993]). These authors price bonds with default risk, options on these risky instruments, and options whose payoffs are subject to credit risk.

My research models the pricing of options on the credit risk of corporate debt, separately from its interest rate risk. This results in a compound option pricing problem where, in a world with stochastic interest rates, the exercise price of the option is stochastic.

Credit risk derivatives are instruments, usually options, whose payoffs are linked to the credit characteristics of a particular asset. Widespread trading of such instruments would enhance the efficiency of the credit markets substantially, and prompt active arbitrage of credit risk separately from term structure risk, which is generally absent from the markets today. Such instruments let fund managers, banks, and corporations increase or reduce their exposure to credit risk, off-balance sheet.

The concept of credit risk derivatives was introduced in 1992 at a conference of the International Swap Dealers Association (ISDA). A number of Wall Street firms have begun experimenting with credit risk derivatives (and Bankers Trust has already begun selling such contracts). The need for such financial instruments cannot be understated. Altman [1991] estimates the default rate for high-yield debt at 6.26% in the first half of 1991. Even insured paper is at risk. For example, in 1991, Moody's downgraded structured securities backed by letters of credit from Credit Suisse.

Credit risk derivatives are not entirely new instruments. Bond insurance, sureties, and financial guarantees constitute a substantial market in the municipal bond area. AMBAC has been insuring municipal bonds since 1971. Yet options on corporate credit risk have, until recently, seen little growth, compared to the municipal market.

There are several possible reasons for this.

Empirical evidence in the muni market clearly shows that the issuer is able to lower the cost of financing with the use of bond insurance (see Hsueh and Chandy [1989] and Jaffe [1992]). It is unclear whether this is true for corporate issuers.

2. There is a very high demand for bond insurance, since municipalities need to raise large amounts of funds, and would find it impossible to do so without backing issues with insurance.[1] Corporate debt issuers have been able to raise funds quite adequately, partly because even junk bonds have a clientele.
3. Suppliers of muni credit insurance have found the business very profitable, as default rates have been less than 0.5% of new issues each year, and capital requirements have been low thus far. On the other hand, corporate debt has evidenced greater default rates.
4. Corporate bond insurance may also be unattractive since the cost of buying insurance is amortized by the issuer over the life of the issue, and it reduces the value of bond callability. Flexibility of capital structure is valuable for corporations, and hence the cost of insurance in terms of reduced callability may outweigh its benefits.

While bond insurance in the muni markets is purchased by the issuer of debt, options on corporate debt are often bought by the holder of debt from investment banks, in specifically structured over-the-counter contracts. As competition grows in the muni bond insurance business, large monoline insurers are entering the market for corporate bond insurance, realizing it is a lucrative market to be tapped.

Wide availability of credit risk options will bring several benefits. If credit risk can be hedged, issuers of risky debt may realize cost savings in financing. Spahr, Sunderman, and Amalu [1991], for example, find that over the period 1970-1985 corporate bond insurance could have been offered by a third-party insurer at lower cost than the prevailing market default risk premium.

As banks hit capital adequacy barriers, they may demand credit risk hedges. Large banks are finding that their positions in swaps and commercial paper are stretching their credit limits. Pension funds will be able to access higher net yields with no additional credit risk. Low-rated corporate issuers will find it easier to access the capital markets, and traders will be able to buy counterparty lines of credit when they need them.

And in recent times credit enhancement has been shown to be a valuable corporate strategy. A case in point is the establishment of AAA subsidiaries by investment banks (see Figlewski [1994]). These subsidiaries will be well-positioned to compete with the large insurers for the corporate bond insurance business. Credit risk will be priced by market forces, and should enhance the efficiency of the credit markets.

Credit risk derivatives may be employed when market players wish to exploit inefficiencies in the market when there is imperfect correlation between stock prices and interest rates. According to my model, for instance, when interest rates and stock prices are negatively correlated, corporate debt values are higher than when the correlation is positive. Credit spreads in the market may not correctly reflect the correlations between stocks and the term structure. This may allow investors to hold a package comprised of a corporate bond and a credit risk derivative, which costs less than equivalent riskless debt, yet offers identical risk/return characteristics.

Municipal bond insurance has been valued so far using actuarial methods. Here instead I present a contingent claims approach to valuing derivatives on credit risk. We will call these derivatives credit risk options (denoted hereafter as CROs). The model provides a framework for pricing default risk, and then describes some of the interesting features of derivatives on credit risk and their hedging. Our valuation methods allow for stochastic asset values and interest rates, and permit any specification of default behavior. The framework is applicable to OTC contracts as well as bond insurance.

The generic form of the CRO (on any underlying risky debt instrument) is a contract in which the writer of the CRO agrees to compensate the buyer for a prespecified fall in credit standing (not necessarily default) of the issuer of the underlying credit instrument. Thus, the CRO protects its buyer against the loss in value of a bond should its yield rise above, or its rating fall below, a trigger or "strike" level. This strike level is usually specified as the acceptable default spread on the bond.

On exercise of the CRO, payoffs are computed as follows: compute the price of the bond at prevailing default-free interest rates plus the strike default spread. This is the "strike price." The payoff of the CRO is the amount by which the strike price exceeds the then-prevailing market price of the bond.[2]

A credit risk option is a compound option with a stochastic exercise price. This is because it is an option on risky debt, which is itself an option on the value of the firm's assets. Given that interest rates are stochastic, so will be the strike price of the CRO.

CROs are different from other options on debt. While debt options usually price the interest rate risk of the bond, CROs price only the credit risk. Therefore CROs strip away credit risk from corporate debt.[3]

Embedded options on credit risk are especially relevant when considering floating-rate contracts where the degree of interest rate risk is less than that of fixed-rate debt. Callable floating-rate notes may be exercised by the issuer when its credit rating improves, and puttable notes may be exercised by the holder when the rating of the issuer declines. These and other similar contracts are termed "step-ups," and are nothing more than options purely on credit risk.

Merton [1974] was the first to suggest a contingent claims approach to pricing risky debt, under the assumption of constant interest rates. The framework we provide for pricing derivatives on risky discount or coupon debt assumes that interest rates are stochastic. Implementation of the methodology requires parameter inputs for the value and riskiness of the firm's assets, which are not directly observable or traded. The model can be used to obtain implied values of these unknown parameters using available security prices.

The behavior of CROs varies dramatically, depending on specific contract terms and conditions, and the model is general enough to accommodate a wide range of such features. For instance, if the CRO is written as a European option, and the firm defaults prior to its maturity, the option value depends very much on whether it kicks in or dies out on default. The model is also able to handle varied specifications of default behavior.

I. THE MODEL

First we review the pricing of CROs in the Merton [1974] framework, where interest rates are treated as constant. For easy reference, the notation used is provided in Exhibit 1.

A recent approach for the pricing of credit risk is the instantaneous default risk approach, where the bond defaults with Poisson probability. Other approaches revolve around modeling the process for credit spreads themselves. While the former seems to be adopted when valuing high-yield bonds, the latter is better for better-quality debt. Yet another approach is to value bonds off the riskless interest rate process and provide for credit ratings to follow a Markov process (Jarrow, Lando, and Turnbull [1994]). The approach I follow allows the specification of stochastic processes for interest rates and firm values, and the discrete time application

EXHIBIT 1
NOTATION

T_B	Maturity of debt
	Maturity of the CRO, $T_W < T_B$
	Value of stock in the firm
	Volatility of stock return
	Face value of debt outstanding
	Market value of debt outstanding
	Value of the firm at time 0
	Firm value at time T_W
	Firm value at time T_B
	Volatility of return on V
	Risk-free interest rate
*	Credit spread for a given debt rating
	Continuous dividend payout rate
	Continuous coupon rate on debt
	B-S call option value on V with strike K, maturity T_W, volatility σ, interest rate r
$P(V, K, T_W, \sigma, r)$	B-S put option value on V with strike K, maturity T_W, volatility σ, interest rate r
K(.)	CRO strike price (stochastic)
$R = R(r, r_d, r^*, r_c)$	Interest rates set

allows imposition of any form of default and boundary conditions.

The maturity of the risky bond occurs at time T_B and that of the CRO at time T_W. Of course, $0 < T_W < T_B$. We assume that the process followed by the value of the firm V is:

$$dV = (\alpha V - r_c F)dt + \sigma V dz \qquad (1)$$

where

α $= r - r_d$;
r_d = rate of continuous dividend payments;
r_c = rate of continuous coupon payments;
F = face value of the debt; and
σ = volatility coefficient of the firm value process.

The solution to Equation (1) for initial value V(0) is:

$$V(t) = V(0)\exp\left[\quad \frac{\sigma^2}{2}\right) t + \sigma Z(t)$$

$$r_c F \exp\left[\quad \frac{\sigma^2}{2}\right) t + \sigma Z(t) \quad \times$$

$$\int_0^t \exp\left[-\left(\alpha - \frac{\sigma^2}{2}\right) s - \sigma Z(s)\right] ds$$

and Z(s), as before, is a standard Wiener process.

Credit options are options on debt for which the underlying stochastic process is not the interest rate but the value of the firm. Hence, in order to compute the CRO value we need to estimate the parameters that define the evolution of the firm value. The value of the firm (V) and the volatility of firm return (σ) are not always ascertainable, but it is possible to back out these values from observable data in a parsimonious way. Alternatively, the models in this article can be used in conjunction with observed prices to infer the necessary parameters.

Valuation of the CRO requires computation of the strike price. This is the value of the bond at time T_W if the credit rating is at the exercise level. Therefore the strike price is the price of a bond discounted at the riskless interest rate plus the strike credit spread (r^*), for the remaining maturity of the bond ($T_B - T_W$). Thus, the effective exercise price (K) is given by

$$K = Fe^{-(r+r^*)(T_B - T_W)}$$

In the case of a coupon bond, the strike price will be the sum of a series of terms of the type above, one for each future cash flow. In that case, F in the expression is replaced by the amount of the future payment, either coupon or principal, and T_B is the payment date.

In this section, where interest rates are assumed to be constant, the exercise price K is also constant. When interest rates (r) are stochastic, so is the exercise price. The payoffs at the maturity of the CRO (i.e., at T_W) are given by:

$$\max(0, K - B_W)$$

where B_W is the value of a bond at time T_W conditional on the fact that the value of the firm at that point in time is V_W. This equation expresses the CRO as a compound option on firm value.

Under the standard put-call parity relationship, we can write

$$B_W = PV(F) - P(V_W, F, T_B - T_W, \sigma, r)$$

$$= V_W - C(V_W, F, T_B - T_W, \sigma, r) \qquad (2)$$

where PV(F) is the present value at T_W of the future bond payments, P[.] is the put option formula of the Black-Scholes equation, and C[.] is the call option formula. We ascertain the value of V_W at which

$$K - V_W + C(V_W, F, T_B - T_W) = 0$$

Call this value V^*. This is the cutoff value of V below which the CRO will be exercised.

For simplicity let us assume a zero-coupon bond and no dividend payout on the equity of the firm; that is, $r_c = r_d = 0$. The CRO value is computed by discounting the expected payoffs taken under the risk-neutral measure. The CRO will be exercised whenever the value of the firm V is less than the cut-off value V^*.

Taking the appropriate integrals, we obtain the expression for the value of the CRO at date 0:

$$CRO(V, \sigma, r, r^*, T_B, T_W, F) =$$

$$e^{-rT_W} \int_{-\infty}^{x[V^*]} [K - Ve^{(r-\sigma^2/2)T_W + x\sigma\sqrt{T_W}}$$

$$C(Ve^{(r-\sigma^2/2)T_W + x\sigma\sqrt{T_W}}$$

$$F,\ T_B - T_W,\ \sigma,\ r)]\, \phi(x)\, dx \qquad (3)$$

where

$$K = F \exp[-(r + r^*)(T_B - T_W)]$$

and

$$x[V^*] = \frac{\log(V^*/V) - (r - \sigma^2/2)\, T_W}{\sigma\sqrt{T_W}}$$

and $\phi(.)$ is the probability density function of the standard normal distribution. Equation (3) is an ordinary integral expression in x, and is easily evaluated using numerical integration.

We can provide a few examples of these computations to demonstrate interaction of the variables in the model. We assume a simple firm with equity S = 25, and debt of face value F = 50 with a remaining maturity of five years (T_B = 5). The risk-free rate is 10% (r = 0.10). The current (instantaneous) volatility of stock return is 35% (σ_S = 0.35). We compute CRO values for various values of the credit risk spread r^*, and the CRO time to maturity (T_W).

CRO prices are computed in two steps. First, we compute the implied values of V and σ using the model with the known values of equity (S) and stock volatility (σ_S). These two parameters map easily into the firm value parameters (V, σ). Second, we apply the valuation for the CRO in Equation (3).

Sample results are portrayed in Exhibit 2, where the values in parentheses are basis points per year.

Results are developed assuming different spreads over AAA rates. For instance in Exhibit 2, assume that AAA paper is trading at the risk-free rate (0 bp over Treasuries).[5] Suppose BAA paper trades at a spread of 100 basis points over AAA paper (it usually does trade 50-150 basis points over AAA). The buyer of the CRO holding AAA paper (of the firm depicted in Exhibit 2), and wanting two-year protection against a fall in its credit spread below that of BAA, would need to pay \$0.117 or 10.5 basis points per year for this protection. The buyer is unprotected against the drop from AAA to BAA, but any further drop in rating is insured against.

In this way, CROs can be used to design floors on credit losses. If the buyer of the CRO is willing to live with a further 50 basis points of deterioration in spreads, another year of protection is possible at the same cost (note that three-year protection at 150 bp also costs 10 bp per year). Hence, such instruments are valuable in matching investor *risk appetite* and investor *holding-period horizons*. Moreover, the cost of additional two-year protection from 150 bp to 100 bp is only 3.5 bp. This seems like a worthwhile trade-off for an investor to make.

These instruments should enhance the effi-

EXHIBIT 2

Simulations of Varying Credit Spreads

T_W (yrs.)	Risk Spread r^* in basis pts. 50	100	150	200
1	0.095	0.026	0.008	0.003
	(17.4)	(5.1)	(1.6)	(0.5)
2	0.202	0.117	0.073	0.047
	(17.0)	(10.5)	(7.0)	(4.5)
3	0.281	0.218	0.175	0.142
	(14.9)	(12.1)	(10.0)	(8.3)
4	0.334	0.303	0.279	0.259
	(12.8)	(11.7)	(11.1)	(10.4)

Computation of CRO values for stock value S = 25. The parameters varied are the time to maturity of the CRO (T_W), and the credit risk spread (r^*). The parameter values are as follows:

Debt amount F = 50
Stock return volatility per year σ_S = 0.35
Riskless interest rate r = 0.10
Average debt maturity (yrs.) T_B = 5

Computed value of the firm is V = 65.46 and value of firm volatility is σ = 0.14. Hence the computed debt-equity ratio is 1.62. Figures in the table are computed CRO values. Figures in parentheses represent CRO premiums in basis points per year.

ciency of the credit markets. The valuation scheme uses parsimonious data, and observable market variables to value CROs. The procedure is also simple, and easy to extend. It has one particularly interesting advantage. Since the model values do not depend on the absolute size of the firm but rather on its debt-equity ratio, *standardized tables* for different values of the debt-equity ratio, stock value, volatility, and time to maturity can be developed and used whenever required.

The value of the CRO falls with a decrease in the debt-equity ratio, because there is more equity to back risky debt. Increases in r^* reduce the CRO value because the effective credit-adjusted exercise price is lower, hence making the option more out of the money. At lower stock values, the debt-equity ratio is higher. For high debt-equity ratios, the CRO premium in basis points per year first increases with increasing T_W, but then decreases, if r^* is low; otherwise, the required basis points per year premium required increases with the period of protection required. Hence, insurance on an annual cost basis is costly for investors with long horizons.

II. MULTIFACTOR MODELS AND STOCHASTIC INTEREST RATES

The analysis so far has assumed a single stochastic process for the value of the firm, which means there is not much flexibility to allow a wide range of default spreads in the model. To enhance the model to encompass stochastic interest rates in addition to stochastic asset values, we can write the firm value process as follows:

$$dV = \mu V dt + \delta_1 V dz_1 + \delta_2 V dz_2 \qquad (4)$$

where $Z = (z_1, z_2)$ is a vector Wiener process. The coefficients defining the variance of this Wiener process are (δ_1, δ_2), and Z is assumed to follow independent Brownian motions.

To allow for stochastic interest rates, we impose the term structure model of Heath, Jarrow, and Morton (HJM) [1992] on the asset process. The law of motion for forward rates is given by

$$df(t, T) = \alpha(t, T)dt + \sigma(t, T)dz_1(t) \qquad (5)$$

where for all $t \leq T$, $f(t, T)$ denotes the instantaneous forward interest rate for forward date T, observed at time t.

Since the firm value process and the term structure of forward rates share the Wiener process z_1, this provides a model where stock prices and interest rates are correlated. This form of the model is discussed in Amin and Jarrow [1992]. The instantaneous spot rate is denoted r(t), and is expressed in terms of the forward rates as $r(t) = f(t, t)$. Default-free bond prices in this model obey the equation:

$$P(t, T) = \exp\left[-\int_t^T f(t, y)dy\right]$$

where P(t, T) is the time t price of a discount bond maturing at time T. Ito's Lemma provides (for a proof, see HJM):

$$dP(t, T) = [r(t) + b(t, T)]P(t, T)dt + a(t, T)P(t, T)dz_1(t) \qquad (6)$$

where

$$a(t, T) = -\int_t^T \sigma(t, y)dy$$

$$b(t, T) = -\int_t^T \alpha(t, y)dy + \frac{1}{2}\left[\int_t^T \sigma(t, y)dy\right]^2$$

To allow risk-neutral pricing of claims in this framework, we start by assuming the existence of a risk-neutral probability measure Q, such that the discounted values of all assets follow martingales. This will ensure pricing using no-arbitrage conditions as shown by Harrison and Kreps [1979]. Since equity (S) is a call option on firm value maturing at time T_B, with a strike price equal to the face value of debt, we can write the value of risky debt in the firm as:

$$B(0, T_B) = V(0) - S(0) =$$

$$E_Q\left[\frac{1}{P(0, T_B)}\right] \quad (7)$$

Detailed computations (along the lines of Amin and Jarrow [1991]) provide the solution to this expression:

$$B(0, T_B) = V(0)[1 - \Phi(k)] -$$

$$FP(0, T_B)\Phi(k - \psi) \quad (8)$$

$$k = \frac{\log[V(0)/FP(0, T_B)] + 1/2\psi^2}{\psi} \quad (9)$$

$$\psi^2 = (\delta_1^2 + \delta_2^2)T_B - 2\delta_1 \int_0^{T_B}$$

$$a(t, T_B)\, dt + \int_0^{T_B} a(t, T_B)^2 dt \quad (10)$$

where $\Phi(.)$ is the cumulative normal density function.

Implementation

To implement the model for various term structures for risky debt, we make some simplifications. They are intended to make computation easier, yet permit a varied set of default spreads.

First, we simplify the volatility process for the term structure by assuming that it stays constant, i.e., $\sigma(t, T) = \sigma$. This makes the term structure model isomorphic to that of Ho and Lee [1986]. Hence:

$$a(t, T) = -\int_0^T \sigma(t, u)\, du$$

$$-\int_t^T \sigma du = -\sigma(T - t)$$

and

$$\int_0^T a(t, T)\, dt = \frac{\sigma T^2}{2}$$

and

$$\int_0^T a(t, T)^2 dt = \frac{\sigma^2 T^3}{3}$$

Second, we assume that the firm has continuous dividend payouts at the rate r_d. With this addition, the value of risky debt is valued using the equation:

$$B(0, T_B) = V(0)[1 - \Phi(k)\, e^{-r_d T_B}$$

$$FP(0, T_B)\Phi(k - \psi)$$

$$k = \frac{\log[V(0)/FP(0, T_B)] - r_d T_B + 1/2\psi^2}{\psi} \quad (12)$$

$$\psi^2 = (\delta_1^2 + \delta_2^2)T_B - \delta_1 \sigma T_B^2 + \frac{\sigma^2 T_B^3}{3}$$

Example

This parsimonious model is capable of generating several risky debt term structures. To illustrate, let us assume that the prices of riskless discount bonds at time 0 satisfy the equation:

$$P(0, T) = \exp[-(0.06 + 0.005T - 0.0001T^2)T] \quad (14)$$

The polynomial assumed above is a simple way of generating a realistic term structure. The instantaneous interest rate is 6% (when $T = 0$). When $T = 1$ the zero-coupon rate is 6.49%. The riskless term structure, embodying the yields $Y(T)$, for different maturities T, is defined by

$$Y(T) = -\frac{1}{T}\log[P(0, T)]$$

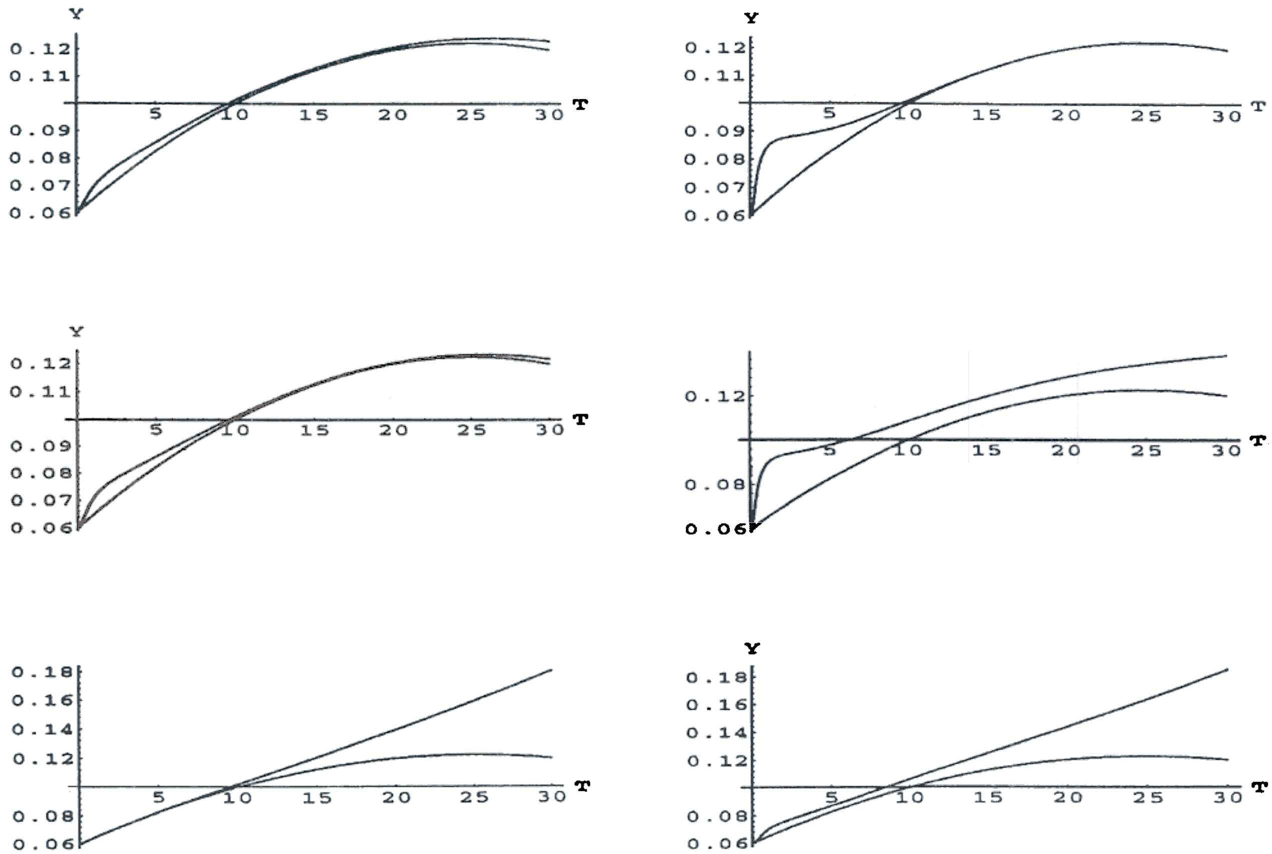
With this initial term structure of interest rates, and suitable parameter values for the processes V and f(t, T), we can compute risky debt prices $[B(0, T_B)]$ for various debt maturities using Equation (11).

The term structure of risky debt with stochastic interest rates, $Y_B(T_B)$, is then expressed as follows:

$$Y_B(T_B) = -\frac{1}{T_B}\log[B(0, T_B)]$$

In Exhibit 3, we plot yield curves $Y_B(.)$ and Y(. against time for different values of parameters.

EXHIBIT 3
Riskless and Risky Term Structures



Yield curves for six different term structures of credit spreads, all using the fundamental parameters: $V(0) = 100$, $F = 75$, $r_d = 0$, $r^* = 0.01$, $T_B = 5$. Reading top to bottom, left to right, the six panels are generated with the following parameter sets for $(\delta_1, \delta_2, \sigma)$: (0.05, 0.2, 0.02), (0.3, 0.01, 0.02), (0.1, 0.2, 0.02), (0.1, 0.3, 0.03), (0.1, 0.1, 0.05), (0.1, 0.2, 0.05). The initial riskless term structure is an upward-sloping one. Note that for ease of reading, the horizontal axis has been placed at a yield of 10%.

The curve $Y_B(.)$ always lies above the curve $Y(.)$ as it is the risky debt term structure. The distance between the two curves represents the default spread for each maturity, and thus essentially represents the term structure of credit spreads.

The model can generate many different shapes for the credit spread term structure. Note in Exhibit 3 that we are able to generate wide spreads at the short end of the term structure, with narrow spreads at the long end; alternatively, we can get narrow spreads at the short end and wide spreads at the long end. The shape of the spread curve is affected by the choice of the three parameters $(\delta_1, \delta_2, \sigma)$. The parameter δ_2, which governs the volatility of fluctuations in firm value that are independent of interest rate movements, affects the spread curve across all maturities. The parameter δ_1, which determines the correlation between changes in firm value and interest rates, impacts spreads only at the short end of the time spectrum, and the parameter σ, which measures yield volatility, impacts only long-maturity spreads.

Derivatives on Risky Debt — Pricing CROs

Pricing options on risky debt is a compound option problem because risky debt itself contains an option on firm value. In our framework, as interest rates are treated as stochastic, we have two components of option value corresponding to the two sources of risk: firm risk and interest rate risk. While some derivatives on risky debt, such as the call feature on corporate bonds, derive value from both sources of risk, credit risk options are specifically written on the firm risk component of the corporate bond stripped of the interest rate risk. Therefore, when pricing credit risk options (CROs), we need to adjust the strike price to reflect this feature.

We first review some of the notation used. As before, the underlying risky bond matures at time T_B, and the CRO on this bond matures at time T_W. Assuming constant firm value volatility, the date 0 forward values for the riskless bond and firm value as of time T_W are as follows:

$$P_W = P(T_W, T_B, Z_1)$$

$$\frac{P(0, T_B)}{P(0, T_W)} \exp\left[\frac{\sigma^2 T_W T_B (T_W - T_B)}{2} - \sigma(T_B - T_W)\sqrt{T_W} Z_1\right]$$

$$V_W = V(T_W, Z_1, Z_2) = V(0) M(T_W, Z_1) \times \exp\left[-\frac{1}{2}(\delta_1^2 + \delta_2^2) T_W + \delta_1 Z + \delta_2 Z_2\right] \qquad (16)$$

where

$$M(T_W, Z_1) = \frac{1}{P(0, T_W)} \times \left[\frac{\sigma^2 T_W^3}{} - Z_1 \sqrt{\frac{\sigma^2 T_W^3}{3}}\right]$$

and (Z_1, Z_2) are the outcomes of random variables (z_1, z_2).

The date T_W price of risky debt for each realization of (Z_1, Z_2) is then given by Equation (11) where the terms $V(0)$ and $P(0, T_W)$ are replaced by the values V_W and P_W from the equations above, and the maturity term T_B is replaced by $T_B - T_W$. We denote this risky debt price using the notation:

$$B[V_W, P_W] = B[T_W, T_B; Z_1, Z_2]$$

We now turn to the computation of the *exercise price* of the CRO. The value of the CRO is that it provides protection to the buyer of the option against a fall in credit rating of the bond below a given level.

As an example, say the current default spread for AA bonds over the riskless rate for a bond of maturity $T_B - T_W$ is 100 basis points. The buyer of the CRO receives compensation in the event that the date T_W price of the risky bond falls below the price of a bond priced at the riskless yield plus 100 basis points. We denote the "strike credit spread" $r^* = 0.0100$. Depending on the outcome of the Wiener process Z_1 as of time T_W, we obtain a different riskless bond price, which translates into a different riskless yield at T_W. Thus, we get a range of strikes $K(.)$ corresponding to each outcome of Z_1. The *stochastic* strike price of the CRO is

$$K(T_W, T_B, Z_1, F, r) = F \times \exp\left(-\left[\frac{1}{(T_B - T_W)} \log P_W + r\right](T_B - T_W)\right)$$

For each outcome of Z_1, we invert the price $P(.)$ to obtain the riskless yield, then add on the spread r^*, and compute the strike $K(.)$. The term in brackets in Equation (17) is the riskless yield plus the strike credit spread r^*. This term is the risky yield, which is then used to compute the exercise price by discounting the face value of the bond for the remaining maturity period.

The price of the CRO is then obtained by taking the necessary expectation under the risk-neutral measure Q:

$$CRO = CRO[\delta_1, \delta_2, \sigma, T_W, T_B, V(0), P(0, .), F, r_d, r^*]$$

$$= E_Q[\max[K(T_W, T_B, Z_1 \quad - B(T_W, T_B), 0]]$$

$$= \int_{Z_1} \int_{Z_2} \max[K(T_W, T_B, Z \quad F, r^*) -$$

$$B(T_W, T_B), 0]\phi(Z_1)\phi(Z_2)dZ_2dZ_1 \quad (18)$$

where $\phi(.)$ is the standard normal density function. The last line follows from the fact that (Z_1, Z_2) are orthogonal.

Computing the integral in Equation (18) is difficult, as it is an extremely complex function involving integrating twice over the cumulative normal distribution function. It also takes considerable computing time. A discrete time approach where the time T_W distributions of Z_1, Z_2 are approximated using a binomial distribution provides very fast and accurate results. For any number n of discrete points used to approximate the probability distribution, we use the following scheme. The ith outcome of the random variable Z (denoted Z_i) takes the value

$$Z_i = \frac{n - 2i}{\sqrt{n}}, \quad \forall i = 0..n$$

with probability

$$Prob_i = \binom{n}{i}\frac{1}{2^n}, \quad \forall i = 0.$$

where $\binom{n}{i}$ = "n choose i," i.e., the number of combinations of n things taken i at a time. It is easily checked that this provides the outcomes of a random variable with mean zero and variance 1 for n steps in the binomial tree. The use of binomial probabilities ensures convergence in the limit to the normal distribution.

The method provides rapid computational speed. For maturities of the order of T_B = five years, even small values of n = 5 provide high levels of accuracy. For longer maturities, n = 30 is sufficient to obtain accurate values, although computing performance is impaired somewhat.

In Exhibit 4, we provide a plot of CRO values for each of the default spread scenarios we developed in Exhibit 3. A value of $r^* = 0.0100$ corresponding to 100 basis points is used. We set $T_B = 5$, and the plots depict CRO values for varying maturity T_W.

Notice that the CROs tend to be valued highest at middle maturities. This is because they are options on options. The value of the CRO includes time value from the underlying bond (which contains an option on firm value) and time value from the option on this bond.

If the CRO maturity is very short, then the maturity of the underlying forward risky bond (which is also an option) is long, and while the underlying instrument has high time value, the CRO itself has very little. The converse occurs when the CRO maturity is long, and the forward risky bond's maturity is short. The highest total time value seems to be attained when both option components have average time value.

The three volatility parameters have different impacts on the prices of CROs. Increases in the riskiness of firm value caused by higher values of δ_1 or δ_2 will increase the value of the CRO. Increase in the volatility of interest rates through the parameter σ causes the value of the CRO to fall. This is because when interest rates are uncertain, the expected discount rate is higher, thus reducing the present value of the CRO.

III. A GENERAL DISCRETE TIME APPROACH

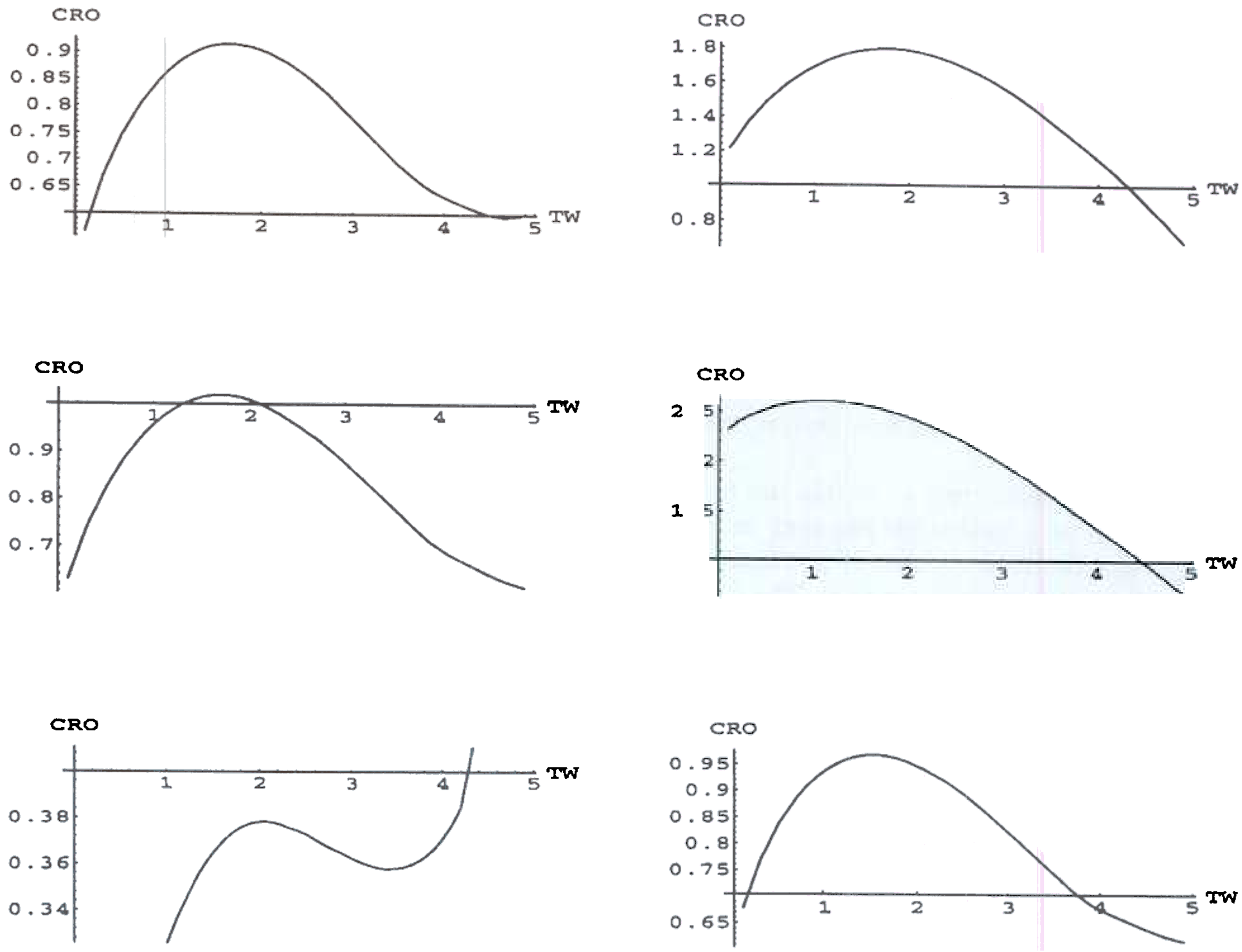
The discrete time model we develop in this section provides far richer features than the continuous time methods. It enables 1) the simple handling of coupon bonds, 2) the pricing of debt with embedded derivatives with American features, 3) an easy modeling of the bankruptcy process as any function of the state variables in the model, and 4) time- and state-dependent volatility functions.[6]

Our model is a discrete time version of HJM augmented for a risky asset process. A full exposition of such a model is provided in Amin and Bodurtha [1994], and the model here is a modest version of their framework. As in the previous section, we write the forward interest rate process in discrete time as

$$f(t + h, T) = f(t, T) + \alpha(t, T)h + \sigma Z_1\sqrt{h}, \quad \forall T \geq t$$

where Z_1 is a standard normal variate, and h is the discrete time interval. Therefore, the instantaneous spot rate is

EXHIBIT 4
PRICES OF CREDIT RISK OPTIONS



CRO values for six different term structures of credit spreads, all using the fundamental parameters: V(0) = 100, F = 75, $r_d = 0$, $r^* = 0.01$, $T_B = 5$. Reading top to bottom, left to right, the six panels are generated with the following parameter sets for $(\delta_1, \delta_2, \sigma)$: (0.05, 0.2, 0.02), (0.3, 0.01, 0.02), (0.1, 0.2, 0.02), (0.1, 0.3, 0.03), (0.1, 0.1, 0.05), (0.1, 0.2, 0.05). The initial riskless term structure is the same as that used in Exhibit 3.

$$r(t) = f(t,t) = f(0,t) + \sum_{j=0}^{\frac{t}{h}-1}\left[\alpha(jh,t)h + \sigma Z_1\sqrt{h}\right]$$

Since σ is scalar constant, this model is similar to that of Ho and Lee [1986]. The price of a riskless bond at time t with maturity T is given by

$$P(t,T) = \exp\left[-\sum_{i=\frac{t}{h}}^{\frac{T}{h}-1} f(t, ih)h\right]$$

$$= \exp\left(-\sum_{i=\frac{t}{h}}^{\frac{T}{h}-1}\left[f(0, ih) + \right.\right.$$

$$\sum_{j=0}^{i-1}\left(\alpha(jh, ih)h + \sigma Z_1\sqrt{h}\right)\Bigg]$$

We define the risky firm's asset process in discrete time as follows:

$$\log\frac{V(t + h)}{V(t)} = [r(t) + \mu(t)]h + \delta\sqrt{h}Z_2(t) \quad (19)$$

where $\mu(t)$ is the excess of the drift over the riskless rate under the risk-neutral measure, and δ is the volatility coefficient. It is assumed here that $\mu(t)$ subsumes an adjustment for risk neutrality, as well as for all continuous payouts of the firm, such as interest and dividend payments.

For simplicity, and to enable the numerical tractability of a path-independent model, we assume that the total payouts of the firm are proportional to firm value at any time. The case where non-proportional payouts are made leads to a path-dependent model, which is no more complex conceptually than the path-independent one, but requires far more computing time. Z_2 is a standard normal variate, and the correlation between Z_1 and Z_2 is denoted ρ.

We assume the existence of a risk-neutral measure under which we can evaluate our random variables Z_1, Z_2. Under the risk-neutral measure, the discounted prices of assets must follow martingales (Harrison and Kreps [1979]), so we compute the drift terms that satisfy the martingale conditions to be

$$\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1} \alpha(t, T)h =$$

$$\frac{1}{h}\log E\left(\exp\left[-h\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1}\sigma Z_1(ih)\sqrt{h}\right]\right) \quad (20)$$

$$\mu(t) = -\frac{1}{h}\log E(\exp[\delta\sqrt{h}Z_2(t)]) \quad (21)$$

By substituting these drifts $[\alpha(.), \mu(.)]$ into the original processes, we obtain the risk-neutral evolution of the firm's risky assets with stochastic interest rates. This transformed process can then be used to carry out valuation of any contingent claim written on the firm value.

This scheme is easy to implement on a bivariate tree. As discussed in Amin and Bodurtha [1994], it is desirable to have path-independence on the tree such that an up move on the tree followed by a down move results in the same value as a down and then up move. This reduces the number of nodes on the tree by a huge order of magnitude, allowing more time steps, and better convergence of the discrete time model to the continuous time one.

The forward rate process above is path-independent (because we assume the special case of the Ho-Lee [1986] model, which is known to be so), but the firm process is not, as it contains r(t) in the drift term (see Equation (19)), which is state-dependent. However, as shown in Amin and Bodurtha [1994], by embedding r(t) instead in the risk term Z_2, and modifying the probability measure, we can attain path-independence.[7]

We transform the correlated random variables Z_1, Z_2 to a pair of orthogonal random variables Y_1, Y_2 for easy implementation. It can be verified that the simple restatement below provides an equivalent risk-neutral pricing process with path-independence.

$$f(t + h, T) = f(t, T) + \alpha(t, T)h + \sigma Y_1\sqrt{h}, \ \forall T \geq t \quad (22)$$

$$\log\frac{V(t + h)}{V(t)} = \mu(t)h + \delta\sqrt{h}\left[\rho Y_1 + \sqrt{1 - \rho^2}Y_2\right] \quad (23)$$

$$m = \frac{r(t)\sqrt{h}}{\delta\sqrt{1 - \rho^2}} \quad (24)$$

where the values of Y_1, Y_2 and probabilities on the bivariate tree are given by

Y_1	Y_2	Probability
−1		
−1	−1	

Under this scheme the expected values and variances and the correlation of Z_1, Z_2 correspond to those of the original system as required. Starting with terminal values for the risky debt in the firm, we compute current values using backward recursion on the bivariate tree.

Since the framework is a discrete time one, it is easy to impose any boundary conditions required. The condition we impose is that whenever the firm value $V(t)$ falls below a certain cutoff level β, bankruptcy will be declared and the debt will be worth only a fraction γ of its face value. Alternative definitions of default may be employed, such as cutoff debt-equity ratios, or any other function of the state variables.

In our numerical example, we impose bankruptcy whenever the value of the firm's assets drops below a value of $\beta = 40$. If this happens, the debt is worth a fraction ($\gamma = 40\%$) of its face value. For illustrative purposes, the debt is assumed to have a maturity of ten years, with a face value of 50, and a coupon of 12%. The initial value of the firm is 100. The stochastic framework for interest rates is that of Ho and Lee [1986], and the initial forward rate term structure is set up by the function:

$$f(0, t) = 0.1 + 0.005t - 0.0001t^2$$

Here the instantaneous forward rate at time 0 is 0.10, and at one year is 0.1049. Interest rate volatility is given by $\sigma = 0.005$, and firm value volatility is $\delta = 0.20$. In Exhibit 5, we plot the risky debt price for different values of the recovery rate γ, interest rate volatility σ, firm value volatility δ, and correlation ρ between firm value and the term structure.

Risky debt prices are naturally increasing in the recovery rate γ. As firm volatility δ increases, risky debt value drops; the same occurs when interest rate volatility σ increases. As the combination of σ and δ interacts through the correlation ρ, increasing correlation naturally reduces the price of risky debt. When correlation between firm values and the term structure is negative, risky debt prices will rise.

We now proceed to value credit risk options on the risky debt values shown in Exhibit 5. For simplicity, we price the CRO in such a way that any fall in credit rating below the risk-free level is compensated for by the option (i.e., strike level $r^* = 0$). The results are depicted in Exhibit 6. The maturity of the CRO is taken to be five years, and the underlying corporate bond is of maturity ten years. The parameters are the same as those used in Exhibit 5. As is expected, the values of the CRO mirror those of risky debt. Here the CRO is assumed to be American in nature; i.e., it may be exercised at any time prior to maturity.

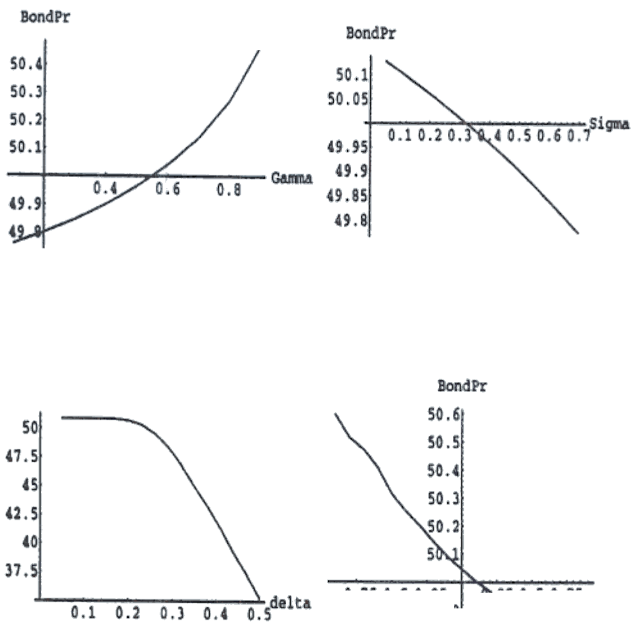
The graphs emphasize some intuitive relationships. The value of the CRO decreases as the recovery rate (γ) increases. As volatility (of assets (δ) or interest rates (σ)) increases, the value of the option (CRO) naturally rises. Finally, as the correlation (ρ) between the changes in the firm's assets and interest rates increases, the value of protection from the CRO increases as well. When the correlation is negative, the volatility from the assets is damped by the offsetting volatility from the term structure of interest rates. As the correlation moves into the positive range, the effect of asset volatility is compounded, leading to higher option values.

IV. HEDGING CREDIT RISK OPTIONS

A natural question is how CROs would be hedged. Since replication is possible, any position in CROs could be notionally offset by a portfolio of stocks and bonds, assuming no short-selling and borrowing constraints. Replication is costly, given transaction costs, however, so writers of CROs could delta-hedge using positions in put options on the firm's stock (which track the CRO better), rather than short-sell the stock. Note that even with put options, rebalancing of the hedge portfolio is still required, making the hedging of CRO portfolios a non-trivial exercise.

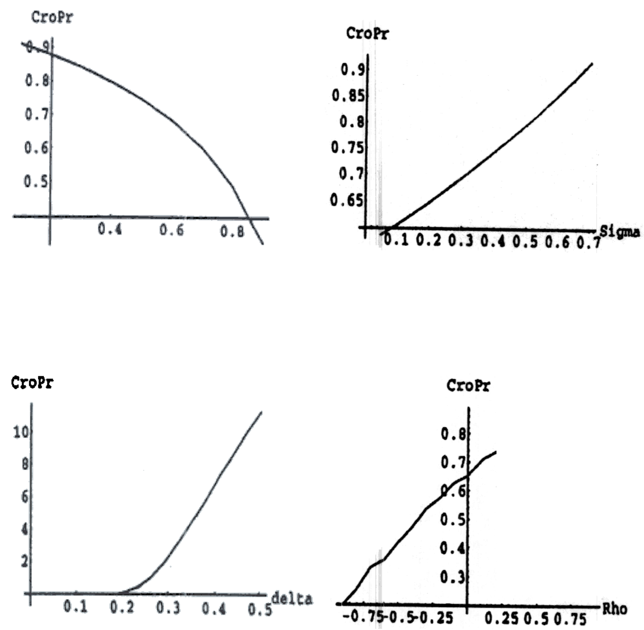
In the theoretical model, bond values are less sensitive to stock values when firms are doing well, and more sensitive when they are doing badly; firms with higher debt-equity ratios display greater CRO sensitivity to changes in firm value and stock value. This matches similar regularities in empirical data.[8]

EXHIBIT 5
PRICES OF RISKY DEBT WITH DEFAULT BOUNDARY IN DISCRETE TIME



Risky debt prices with the parameters: firm value = 100, debt face value = 50, coupon = 12%, $\sigma = 0.005$, $\delta = 0.25$, correlation between interest rates and firm value = 0.5, debt maturity = 15 years, recovery rate on default = 40%. The equation for the initial forward rate curve is $f(t) = 0.1 + 0.005t - 0.0001t^2$. Graphs are plotted for various values of γ, σ, δ, and ρ. σ on the graph is shown times 100.

EXHIBIT 6
PRICES OF CREDIT RISK OPTIONS WITH DEFAULT BOUNDARY IN DISCRETE TIME



CRO prices with the parameters: firm value = 100, debt face value = 50, coupon = 12%, $\sigma = 0.005$, $\delta = 0.25$, correlation between interest rates and firm value = 0.5, debt maturity = 15 years, recovery rate on default = 40%. The equation for the initial forward rate curve is $f(t) = 0.1 + 0.005t - 0.0001t^2$. The maturity of the CRO is five years. Graphs are plotted for various values of γ, σ, δ, and ρ. σ on the graph is shown times 100.

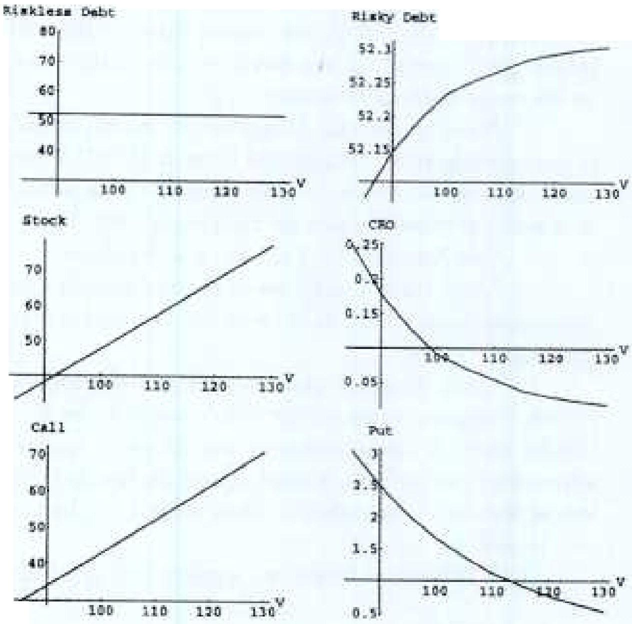
There are two sources of risk in the CRO: the riskiness of the firm's assets and the riskiness of interest rates, meaning that at least two hedge instruments will be required. In Exhibits 7 and 8, we plot the values of risky debt, risk-free debt, the CRO (American), stock in the firm, and call and put options (European) on the stock, when there is default risk. In Exhibit 7 values are plotted for varying firm value V, and in Exhibit 8 for varying levels of the instantaneous interest rate r_0.

The graphs indicate which corporate securities best mimic the behavior of the CRO. For instance, notice that the put option value closely tracks that of the CRO for changes in firm value. This suggests that buying puts may provide a simple way to hedge the exposure of the CRO to variations in firm value.

In Exhibit 8, with values plotted for varying interest rate levels, once again the put option value tracks that of the CRO well. Yet it should be clear from the graphs in Exhibits 7 and 8 that the CRO is by far the most convex of all the securities. This implies more frequent hedge rebalancing will be required, imposing greater transaction costs.

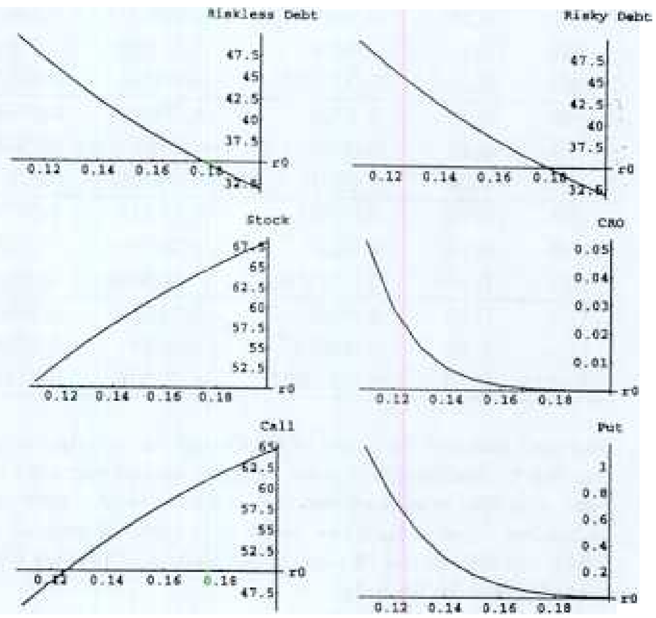
One simple way of hedging the CRO is to use a put option and riskless debt in some combination to eliminate both firm risk and interest rate risk in an instantaneous fashion. Choose the number of puts (denoted N_p) such that the entire sensitivity of the CRO to firm risk is removed, and then use a certain amount of riskless debt (N_b) to eliminate the remaining interest rate risk. The hedge must satisfy the equations:

EXHIBIT 7
PRICES OF CORPORATE SECURITIES WITH DEFAULT BOUNDARY IN DISCRETE TIME (FOR VARYING FIRM VALUE)



Prices of various corporate securities with the parameters: firm value = 100, debt face value = 50, coupon = 12%, $\sigma = 0.005$, $\delta = 0.20$, correlation between interest rates and firm value = 0.5, debt maturity = 10 years, recovery rate on default = 40%. The equation for the initial forward rate curve is $f(t) = 0.1 + 0.005t - 0.0001t^2$. The maturity of the CRO, calls, and puts is five years. The strike level of the CRO is $r^* = 0$, and that of the calls and puts is 50. The CRO is American-style, and the calls and puts are European. The default boundary is at V = 40.

EXHIBIT 8
PRICES OF CORPORATE SECURITIES WITH DEFAULT BOUNDARY IN DISCRETE TIME (FOR VARYING INTEREST RATES)



Prices of various corporate securities with the parameters: firm value = 100, debt face value = 50, coupon = 12%, $\sigma = 0.005$, $\delta = 0.20$, correlation between interest rates and firm value = 0.5, debt maturity = 10 years, recovery rate on default = 40%. The equation for the initial forward rate curve is $f(t) = r_0 + 0.005t - 0.0001t^2$. The maturity of the CRO, calls, and puts is five years. The strike level of the CRO is $r^* = 0$, and that of the calls and puts is 50. The CRO is American-style, and the calls and puts are European. The default boundary is at V = 40.

$$\frac{\partial CRO}{\partial V} = N_p \frac{\partial PUT}{\partial V}$$

$$\frac{\partial CRO}{\partial f} = N_p \frac{\partial PUT}{\partial f} + N_b \frac{\partial BOND}{\partial f}$$

These equations may be solved for (N_p, N_b) to obtain the local hedges.

In Exhibit 9, we present a numerical example of the required holding of puts and risk-free debt needed to hedge the CRO. The analysis is carried out for a range of firm values and interest rates. The values in the table provide the CRO price, and the amount of the hedge required.

Hedging with puts and risk-free debt is the simple way to make up the correct hedge. This follows immediately from the fact that risky debt is valued as the difference between a riskless bond and a put on firm value. Similarly, a derivative written on risky debt shares the same characteristics. A similar hedging approach is suggested by Chance [1990].

V. CONCLUDING COMMENTS

The stochastic interest rate models for the pricing of derivatives on risky debt appear to allow for all plausible shapes of default spread curves. CROs enable splitting away and pricing the risk of default only.

Firm A can buy a CRO from Ba

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