Pricing Credit-Sensitive Debt when Interest Rates, Credit Ratings and Credit Spreads are Stochastic

Sanjiv Ranjan Das*
Peter Tufano
Harvard Business School

First Version: December 1994
This Version: September 1995

Abstract

This paper develops a model for the pricing of credit-sensitive debt contracts. Over the past two decades, the debt markets have seen a proliferation of contracts designed to reapportion interest rate and credit risks between issuer and investors. Contracts including credit-sensitive notes (CSNs), spread adjusted notes (SPANs), and floating rate notes (FRNs) adjust investors' exposures to three risks: interest rate risk, changes in credit risk caused by changes in the credit rating of the issuer of the debt, and changes in credit risk caused by changes in spreads on the debt, even when ratings have not changed. In the paper, we develop a pricing model incorporating all three risks, with special emphasis on credit risks. The model incorporates a decomposition of credit spreads into two stochastic elements: the default process and the recovery process in the event of default. The model is easily implementable as it uses observable inputs. By using a discrete time formulation the model is numerically easy to employ, and also permits the pricing of debt with embedded options of American type. It also allows for pricing contracts between parties with varying credit ratings such as swaps where each counterparty may have different credit quality. These features impart a degree of generality and practicality to the model which should make it attractive to academics and practitioners alike.

---

*Preliminary, comments invited. We would like to thank Edward Altman, Fisher Black, John Campbell, George Constantinides, Greg Duffee, Alan Eberhart, Jerome Fons, Silverio Foresi, Ravi Jagannathan, Scott Mason, Robert Merton, especially Jesus Saa-Requejo, and participants at the NBER Finance lunch group, the Finance faculty at Santa-Clara University, the Hong Kong University of Science and Technology, Boston University, Federal Reserve Cleveland, IAFE conference (New York 1995), and the NBER conference on Risk and Derivatives for helpful discussions. Address all correspondence to the authors at Morgan Hall, Graduate School of Business Administration, Harvard University, Soldiers Field, Boston, MA 02163, Tel: 617-495-6080/6855.

Pricing Credit-Sensitive Debt when Interest Rates, Credit Ratings and Credit Spreads are Stochastic

This paper develops a model for the pricing of credit-sensitive debt contracts. Over the past two decades, the debt markets have seen a proliferation of contracts designed to reapportion interest rate and credit risks between issuer and investors. Contracts including credit-sensitive notes (CSNs), spread adjusted notes (SPANs), and floating rate notes (FRNs) adjust investors' exposures to three risks: interest rate risk, changes in credit risk caused by changes in the credit rating of the issuer of the debt, and changes in credit risk caused by changes in spreads on the debt, even when ratings have not changed. In the paper, we develop a pricing model incorporating all three risks, with special emphasis on credit risks. The model incorporates a decomposition of credit spreads into two stochastic elements: the default process and the recovery process in the event of default. The model is easily implementable as it uses observable inputs. By using a discrete time formulation the model is numerically easy to employ, and also permits the pricing of debt with embedded options of American type. It also allows for pricing contracts between parties with varying credit ratings such as swaps where each counterparty may have different credit quality. These features impart a degree of generality and practicality to the model which should make it attractive to academics and practitioners alike.

1 Introduction

This paper develops a model for the pricing of a wide range of risky debt contracts and derivatives, including those whose terms are contingent on credit ratings. Over the past two decades, the debt markets have witnessed a proliferation of contracts aiming to reallocate interest rate and credit risk between issuers and investors. In the mid-1970s, new contracts that modified investors' interest rate exposure included various forms of floating rate notes (FRNs). In the late-1980s, security designers attempted to modify investors' exposure to credit risk with event-risk protection. These experiments typically stated that a triggering event (such as a ratings downgrade) would either give the investor the right to put the debt back to the firm (event risk covenants) or enjoy increased coupon payments (credit-sensitive debt). Finally, in the early 1990s, financial intermediaries began to sell credit-risk derivatives whereby investors could modify existing credit exposures, just as other derivatives permitted investors to modify the interest rate exposures of already-issued instruments. For example, a financial intermediary might offer options on the credit spread of a particular issuer or a ratings class (such as AAA corporates) against a reference Treasury.

Innovations in the new issue and derivative markets that are credit-rating-contingent (event risk covenants, credit-sensitive debt, and credit risk derivatives) demand that credit ratings be explicitly incorporated into pricing models. Yet fixed income pricing models that explicitly reference credit ratings are useful for much more than pricing exotica. A number of researchers have recognized that credit ratings and credit spreads provide useful observable data upon which pricing models can be based that can value a wide range of fixed income instruments. By definition, credit ratings, which are easily observed, published by a handful of rating agencies, and carefully scrutinized by investors, reflect an assessment of the likelihood of default. Credit spreads provide additional information regarding the expected recovery rate in the event of default, as well as the market's credit risk premium.

The literature on pricing instruments with credit risks has followed two related approaches:

- Models which assume a stochastic process for the value of the firm, and then treat risky debt as an option (Merton [17], Longstaff and Schwartz [14] Black and Cox [5], Shimko, Tejima and VanDeventer [21], and Nielsen, Saa-Requejo and Santa-Clara [18]).

- Models which assume stochastic processes for (a) the credit quality of each bond (Jarrow, Lando and Turnbull (JLT) [13]) and (b) the recovery rate in the event of default.

In general, models for pricing risky debt can be expressed simply using the following equation:

$$P(r, t, .) = B(r, t) - \delta(.)Q(.)B(r, t) \tag{1}$$

where r is the riskless interest rate, t is maturity, $P(.)$ is the price of zero coupon risky debt, $B(.)$ is the price of riskless debt of the same maturity, $Q(.)$ is the psuedo-probability of default and $\delta(.) = 1 - \beta(.)$ is equal to one minus the amount of the bond's value recovered in the event of default. As we shall see later, $\beta(.)$ is the recovery rate on default.

Models that treat the value of the firm as the underlying stochastic process write $\delta(.)$ and $Q(.)$ as functions of firm value and the debt claims issued by the firm. While this approach is well-grounded in theory, it has the practical difficulty of being predicated on a difficult to observe stochastic process, the value of the unlevered firm. Therefore, the second approach has the potential to treat $Q(.)$ and $\delta(.)$ as stochastic processes, utilizing the information about these functions that is embedded in observed credit spreads and recovery rates, such as in JLT.

Our model is an extension of JLT, and like that model relies on an observable bond trait, credit ratings, to characterize the probability of default. In their model and ours, the quality level of a bond passes from one level to another under a probablity law depicted by a Markov chain. One of these quality levels or states is that of default. In the JLT model, on default, the bond offers a specific payoff governed by a *constant* recovery rate. Therefore, the JLT model is characterized by a variability in spreads which is driven purely by changes in credit ratings. We enhance their model by making the recovery rate in the event of default stochastic as well, thus providing a two-factor decomposition of credit spreads. This has the following impact:

- One, it allows more variability in the spreads on risky debt. The standard Merton model for risky debt does not appear to successfully generate the magnitude of credit spreads usually observed in the market. By permitting the recovery rate to be stochastic, our model generates spreads that are closer in magnitude and variablility to those seen in the debt markets.

- Two, spreads are now a function of factors other than pure quality levels. In the Jarrow-Lando-Turnbull model credit spreads change only when credit ratings change, whereas in the debt markets we find that credit spreads change even when ratings have not changed. Injecting stochastic recovery rates into the model provides this extra feature.

- Three, we choose to make recovery rates correlated with the term structure of interest rates. This results in a model wherein credit spreads are correlated with interest rates, as is evidenced in practice.

- Four, in the Jarrow-Lando-Turnbull model (which is a model to explain average spreads levels for each rating class), the debt of all firms in the same rating class will demonstrate identical variability of credit spreads. In our model, by choosing different recovery rate processes for firms within the same credit rating class, we are able to generate variability of spreads which is firm specific, rather than rating class specific.

- Finally, making recovery rates stochastic enables the pricing of a wide range of spread-based exotic debt and option contracts. Our model provides a means to price resetting debt when the yardstick for the reset may be the riskless interest rate, the firm's credit rating, or its spread over Treasuries. This permits us to value risky debt when the counterparties have different credit risks, for example, in the pricing of risky coupon swaps. Our model's explicit link to observable credit ratings and credit spreads is critical. Not only does it make the implementation of the model feasible, but also it permits us to value standard bonds, and credit-contingent instruments.

In section 2, we present stylized facts about interest rates, credit spreads, and recovery rates that underlie our model. In section 3, we develop the pricing model. In section 4, we use the model to price different debt claims, presenting illustrative examples of pricing, as well as a more detailed discussion of the pricing of a particular credit-sensitive note. Section 5 concludes.

2 An Empirical Overview

According the the published definitions of rating categories given by the major rating agencies, credit ratings primarily reflect the *likelihood of default.* For example, Standard and Poor's notes that "likelihood of default is indicated by an issuer's senior debt rating," and goes on to define the various rating categories by using language reflecting the firm's "capacity to pay interest and repay principal." The rate that might be recovered in the event of default is at most a secondary consideration; for example, Standard and Poor's explains that all firms in default receive D ratings, regardless of their outlook (or presumably the expected recovery rates bondholders might expect). Ratings are an easily observable trait in that most firms are rated by one of the major agencies. For example, Standard and Poor's rates approximately 4000 domestic and international corporate issuers. We exploit the breadth of ratings and their institutional definition to provide a useful characterization of the probability of default.

Credit spreads for a particular rating category reflect not only expectations of the probability of default, but also expectations of recovery rates. Credit spreads by rating category are reported in the press, and firm-specific credit spreads can be calculated from the prices of the firm's traded obligations. Because defaults are relatively rare, meaningful recovery rates can be calculated only on a market-wide basis. Recovery rates are typically reported as the percentage of a defaulted bond's principal value received by the bondholder following a bankruptcy or reorganization proceeding. Credit ratings are increasingly being used as an indicator of credit quality. Lucas [15] reports that more and more debt issues contain downgrade trigger clauses for determining coupon and collateral levels.

Because our model uses credit ratings, default probabilities, credit spreads, and recovery rates extensively, it is appropriate to summarize stylized empirical facts about these traits of the fixed income market, as well as the relationships among them. Specifically, we sketch empirical relationships between the term structure, risky debt yields, credit spreads and recovery rates on risky debt. Our data consists of monthly observations, from 1976 through 1991, of the yield on constant-maturity Treasuries of maturity 2,5,10 and 30 years and the yield on corporate bonds for four credit rating levels: AAA, AA, A and BAA. Spreads for all four rating levels are computed as the yield spread over the 30 year constant-maturity yield.

The data was obtained from Moody's Bond Service. Recovery rate data for four classes of debt (secured, senior, senior-subordinated and subordinated), is obtained from Altman's studies [1] of risky debt. We also refer to Duffee's [8] recent work, which provides more extensive evidence about default risk and Treasury yields.

Table 1 presents descriptive statistics for Treasuries, corporate bonds, and credit spreads. Note that the standard deviation of Treasury yields increases with maturity, whereas that of the corporates increases with a fall in credit rating. Spreads also show greater variability for lower-rated bonds. Duffee finds similar results in his examination of individual bonds from the Lehman Brothers dataset from 1985 to 1994.

Table 2 presents the correlations between Treasury yields and corporate bond yields. Table 3 presents the correlation of the spreads with Treasury yields as well as with other spreads. AAA spreads show little correlation with the level of riskless rates, whereas BAA spreads appear to have a strong positive correlation with the level of Treasury yields. We also see that credit spreads are correlated across rating categories, but this relationship is strongest for nearby credit ratings

Table 4 presents the correlation of changes (first differences) in spreads with those of the changes in other variables in our study. Using aggregate data, changes in spreads for all but AAA issuers appear to be weakly negatively correlated with contemporaneous changes in Treasury yields. Using firm-level data and controlling for bond maturity, Duffee finds a stronger result. In his sample, he finds a statistically significant negative relationship between changes in credit spreads (measured relative to the appropriate Treasury) and contemporaneous changes in equivilent-maturity Treasury yields, expecially for lower-rated issues. As might be expected, changes in spreads are strongly correlated between adjacent credit ratings.

Table 5 presents the relationship between credit spreads and the recovery rates on defaulted debt. Recovery rates, taken from Altman [1] reflect the percentage of face value received by bondholders when a bond defaults. In our model, we relate the variability of credit spreads to the variability in expected recovery rates. In particular, increasing recovery rates should result in decreasing spreads as the capital at risk is reduced. The evidence in Table 4 suggests that this relationship is borne out in the data: contemporaneous credit spreads and recovery rates are strongly negatively correlated. Furthermore, this correlation is strongest between lower-rated issues and the least senior debt issues. However, Table 5 represents limited data (annual observations for the period 1985-91), and though it bears out what is a simple theoretical proposition, no conclusive empirical implications can be drawn. Because spreads are in turn correlated with interest rates from Table 2, this gives us reason to allow for a correlation between recovery rates and interest rates. However, our data here is limited and consists of simply annual observations from 1985-91, and as such, much more empirical work is required on the observed behavior of recovery rates in order to make definitive assumptions. Meanwhile, our model provides a normative stance on the behavior of recovery rates, and their impact on credit spreads.

Rating agencies routinely publish statistics on rating changes, or the transition probabilities of moving from one rating to another. Duffee analyzes whether these transition probabilities are related to changes in 10-year Treasury yields. For financial service firms, utilities (except those rated A), and industrial issuers (except those rated AA and A), he found no significant relationship between changes in Treasury yields and one-year transition probabilities. For AA-rated industrials and A-rated industrials and utilities, he found significant relationships with interest rate changes. For the industrials, increases in Treasury rates were associated with upgrades.

Our current model allows us to incorporate many of these stylized features of the credit market. Specifically, we generate stochastic credit spreads by incorporating stochastic recovery rates, which are correlated with the level of riskless rates. A firm's credit rating (or probability of default) is also stochastic with changes in ratings are represented by the market-wide transition probabilities. In the current model, the statistical transition probability matrix is constant; but, as we shall see, the risk-neutral transition matrix will be time varying.

3 Model

The pricing model for CSNs falls into two segments: one, the term structure model, and two, the default model. For the term structure model, we adopt the one factor Heath-Jarrow-

Morton ([10]) (HJM) model in discrete time. This is a parsimonious model as it requires just one parameter, volatility. It can also be used to match the existing term structure of zero coupon bonds.

Default is parameterized as a combination of the probability of default and the recovery rate in the event of default. As mentioned earlier, the probability of default is given by a bond's rating, and the recovery rate is reflected in the credit spread for that rating. Therefore, we will use the terms "default risk" and "credit rating change risk" synonomously, and the terms "recovery rate risk" and "spread risk" interchangeably.[1] The model considers both default risk and spread risk. For the default model we assume that the credit rating of risky debt follows a Markov chain. The probabilities of moving from one credit rating to another are specified by a transition probability matrix, which is easily estimated. This model has been posited by Jarrow, Lando and Turnbull ([13]) (JLT), and is easily adapted to suit our valuation goals. Thus, our modeling approach involves an amalgam of existing models, as well as an extension to stochastic default recovery rates, as in Jarrow and Turnbull [12], which may be correlated with the term structure of interest rates.

Our modelling approach is a standard contingent claims one. The two components are the term structure model and the default risk model. We shall obtain the risk-neutral probabilities for the evolution of the term structure of interest rates, and then ascertain the risk-neutral probabilities of the default process. The combination of the two provides a stochastic framework for the arbitrage-free pricing of risky debt.

3.1 The Term Structure Model

Our model is a discrete time version of the Heath-Jarrow-Morton [10] (HJM). A full exposition of this type of model is provided in Amin & Bodurtha [3], and the model here is a modest version of their framework. We write the forward rates process in discrete time as

$$f(t+h,T) = f(t,T) + \alpha(t,T)h + \sigma(t,T)X_1\sqrt{h}, \;\; \forall T \geq t$$

[1] Normally, spreads are functions of both the probability of default and the recovery rate. Here we employ a restricted definition of spread risk, denoting spreads conditional upon no change in the probability of default.

where X_1 is a standard Normal variate and h is the discrete time interval. $f(t,T)$ is the one period forward rate at time t for one period starting at time T, $\alpha(t,T)$ is the time varying drift term for the forward rate process, and $\sigma(t,T)$ is the volatility coefficient. Therefore, the instantaneous spot rate is

$$r(t) = f(t,t) = f(0,t) + \sum_{j=0}^{\frac{t}{h}-1} [\alpha(jh,t)h + \sigma(jh,t)X_1\sqrt{h}]$$

When σ is scalar constant, this model exactly mimics that of Ho & Lee [11]. The price of a zero-coupon bond paying \$1 at time t with maturity T is given by

$$\begin{aligned} P(t,T) &= \exp\left[-\sum_{i=\frac{t}{h}}^{\frac{T}{h}-1} f(t,ih)h\right] \\ &= \exp\left(-\sum_{i=\frac{t}{h}}^{\frac{T}{h}-1}\left[f(0,ih) + \sum_{j=0}^{i-1}(\alpha(jh,ih)h + \sigma(jh,ih)X_1\sqrt{h})\right] h\right) \end{aligned}$$

Defining a riskless money market account $B(t)$ as the numeraire for pricing bonds, we can write:

$$\begin{aligned} B(t) &= \exp\left[\sum_{i=0}^{\frac{t}{h}-1} r(ih)h\right] \\ &= \exp\left(\sum_{i=0}^{\frac{t}{h}-1}\left[f(0,ih) + \sum_{j=0}^{i-1}(\alpha(jh,ih)h + \sigma(jh,ih)X_1\sqrt{h})\right] h\right) \end{aligned}$$

We assume the existence of a risk-neutral measure under which we can evaluate our random variable X_1. We also assume the market for riskless and risky interest rate claims is complete. Define

$$Z(t,T) = \frac{P(t,T)}{B(t)}$$

Under the risk-neutral measure, the discounted prices of assets must follow martingales (Harrison & Kreps [9]), and so

$$E\frac{Z(t+h,T)}{Z(t,T)} = 1 \tag{2}$$

where E denotes the expectation taken under risk-neutrality at time t. (Throughout the rest of the paper, we shall assume that $E(.) \sim E_t(.)$). Making the necessary substitutions into the no-arbitrage condition (2) above, we can solve for the values of the drifts that satisfy this condition:

$$\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1} \alpha(t,ih)h = \frac{1}{h}\log E\left(\exp\left[-h\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1}\sigma(t,ih)X_1(t)\sqrt{h}\right]\right) \qquad (3)$$

By substituting the drift $[\alpha(.)]$ into the original process, we obtain the risk-neutral evolution of the term structure. This transformed process can then be used to carry out the valuation of any contingent claim written on stochastic interest rates. Such a tree makes numerical implementation economical and practical. Another specification with mean reversion is $\sigma(.) = \sigma\exp[-\lambda(T-t)]$. Here, we obtain a Gaussian interest rate process with mean reversion at rate λ, akin to that specified by Vasicek [26]. This is the process used in the paper. Other forms may also be used, and to the extent they are not dependent on the state variables $f(.)$, we can attain a path-independent implementation. Because we model changes in discrete time, it is easy to impose any type of boundary conditions called for by embedded option features in the risky debt product.

3.2 The Default Model

The default segment of the model consists of two parts: the default process and the recovery rate process. The default process is similar to that adopted in Jarrow-Lando-Turnbull [13], while the recovery rate process comprises the innovation of the paper.

3.2.1 *Default Process:*

As in Jarrow, Lando and Turnbull [13], we assume that the credit rating of the bonds follows a discrete-time, discrete-space Markov chain. As reported in the Standard and Poor's Credit Review, we assume that there are eight possible credit rating levels (AAA, AA, A, BBB, BB, B, C, and D), although, we may specify any number of rating levels. We index the rating levels by $i = 1..K$. In our specific case, $K = 8$. The rating D ($i = 8$) stands for the default category. The transition probability matrix for moving from state i to state j in time interval h is denoted P, and is depicted below:

$$P(h) = \{p_{ij}\}_{i,j=1..K} = \begin{bmatrix} p_{11} & p_{12} & .. & .. & .. & .. & .. & p_{18} \\ p_{21} & p_{22} & .. & .. & .. & .. & .. & p_{28} \\ : & : & : & : & : & : & : & : \\ : & : & : & : & : & : & : & : \\ p_{71} & p_{72} & .. & .. & .. & .. & .. & p_{78} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

The last row in this matrix indicates that when the bond passes into default it stays in default, i.e. the state is absorbing. This matrix P is obtained using historical data on rating changes and defaults. The transition matrix over the period nh is simply P^n, which follows from an elementary result (the Chapman-Kolmogorov equations) in the theory of Markov chains. The matrix P^n represents the n-period cumulative probability of default.

For each of the eight states, the bond markets assign maturity-specific spreads over the risk free rate at which risky bonds will be traded.[2] For instance, at a rating of AA, the market may require a spread of 200 basis points for a bond of maturity 5 years. The term structure of forward credit spreads for each of the eight ratings is thus obtained from market data. Denote $s_i(T)$ as the credit spread on a T-maturity forward one period bond of current rating i. The term structure of forward credit spreads is specified by the spread matrix $\{s_i(T)\}_{i=1..K,t<T}$.

The matrix P above is the statistically observed transition probability matrix. For the pricing of risky debt to satisfy no-arbitrage conditions, we need to employ the risk-neutral transition probability matrix. We assume the existence of a transition probability matrix Q, which we shall denote as the equivalent probability measure matrix. Under the Q measure, assume that the prices of risky bonds follow a martingale. Since Q is an equivalent measure we require that any cell $\{i,j\}$ in Q must be zero for any cell in P that is zero. It is assumed (as in Jarrow, Lando and Turnbull [13]) that the elements of Q, $q_{ij}, i, j = 1..K$, bear the following relationship to the elements of the P matrix:

$$q_{ij} \quad = \quad \pi(i)\, p_{ij}, \quad \forall i \neq j$$

[2] In reality, the full term structure of credit spreads is rarely available. The Wall Street Journal reports a given spread for maturities out to 10 years, and another one for longer maturities.

$$
\begin{aligned}
q_{ii} &= 1 - \sum_{j=1..K, j \neq i} q_{ij} \\
& 0 \leq q_{ij} \leq 1, \quad \forall j
\end{aligned} \tag{5}
$$

It can be easily shown that the relationship between the P and Q matrices is given by the following expression:

$$
Q = I + \mathrm{Diag}[\Pi](P - I) \tag{6}
$$

where Π is the vector of $\pi(i), i = 1..K$. The terms $\pi(i)$ may be thought of as a risk adjustment for default risk premia. On account of the multiplicative relationship of Π between P and Q, we are able to ensure that the measures are equivalent, as sets of measure zero under P will also be sets of measure zero under Q.

Before we proceed to compute the elements of Q we introduce a simplifying assumption, that is, the process driving the term structure and that driving the default process are independent. Therefore the values in the matrix P are not functions of $f(t,T), \forall T > t$. We also note that whereas the matrix P is time homogenous, the matrix Q is not, as the risk premium adjustment $\pi(i)$ will be a function of time. Therefore we obtain a series of Q matrices, $Q(t), t = 1..T$.

Let us define a stochastic cashflow vector $C \in R^K$, which provides us the payoffs on a one dollar zero coupon bond at its maturity in each state $i, i = 1..K$. Standard and Poors adopt 8 rating levels: AAA, AA, A, BBB, BB, BBB, C and D where D is default. C will be written as

$$
C = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \tilde{\beta} \end{pmatrix}
$$

The vector above shows that in all states except default ($i = 8$, state D), the bond pays off its promised amount, one dollar. In state D, the bond only pays off $\tilde{\beta}$, which is the *stochastic* recovery rate on the defaulted bond, given that it defaulted prior to maturity. It is not necessary for the bond to default at maturity only, as the Markov chain computes the cumulative probability of default upto maturity. We assume here that if default occurs, the bondholder will receive an amount β, the recovery rate. This recovery rate is stochastic and is correlated with the spot rate process. Making the recovery rate process correlated with the term structure enables us to inject a macroeconomic influence on default spreads within the model.

3.2.2 *Stochastic Recovery Rates:*

The stochastic recovery rate determines the level of credit spreads in each period. This is because the forward credit spread must be a function of the amount recovered on default in the specific forward period. Thus, we establish a simple link between stochastic spreads and recovery rates. The stochastic recovery rate also provides the link between our model and that of the Merton type, because recovery rates are directly linked to the value of the firm.

Permitting recovery rates to be stochastic in our model is important for the following reasons:

1. Credit spreads fluctuate when either (i) the probability of default changes or (ii) the ex-ante recovery rate conditional on default changes. The model therefore incorporates a two-factor stochastic model for changes in credit spreads. Therefore, when the recovery rate is stochastic, credit spreads may change even when the credit rating of the firm has not. This is clearly realistic.

2. In a model where only credit ratings change but not recovery rates, all firms within the same rating class will demonstrate identical variability in credit spreads. In our model, by choosing different recovery rate processes for individual firms, we allow for a wide range of spread behavior within the same rating class.

3. Making recovery rates stochastic injects additional variability into the model spreads. This is critical as models such as the standard Merton model have been found to generate smaller and less variable spreads than those observed in practice.

4. Empirically, credit spreads are found to be correlated with the term structure of interest

rates (see Tables 3-5). This may be because both the probability of default and the recovery rate on default may depend on existing macroeconomic conditions. It is sufficient to make one of the above two factors correlated with interest rates. To see this, notice that analogous to equation (1), the price of a credit-sensitive bond is equal to the price of a default-free bond less an adjustment for default risk. Thus,

$$P(r, X, t) = B(r, t) - (1 - \beta(r, X, t))Q(r, X, t)B(r, t)$$

where both the recovery rate $\beta(.)$ and the default probability $Q(.)$ depend on the interest rate r and an additional state variable X, which accommodates the riskiness of the debt. The term $(1 - \beta(.))Q(.)$ is responsible for the magnitude and variability of credit spreads. In order for spreads to be correlated with the term structure of interest rates (r), clearly there is no need for both β and Q to be correlated with r, and the spread term could be keyed off the composite term $\beta(r, t)Q(X, t)$. This approach greatly simplifies the model analytics while retaining a key empirical feature.

5. Finally, stochastic recovery rates enable the pricing of several new forms of risky debt such as spread adjusted notes and various kinds of spread options.

We are now in a position to establish a bivariate process in spot rates and recovery rates. Assume that the initial recovery rate is $\beta(0)$. We also assume that recovery rates obey the following stochastic process:

$$\beta(t+h) = \left(1 + \frac{1-\beta(t)}{\beta(t)} \exp(\sigma_\beta X_2 \sqrt{h})\right)^{-1}$$

It is easily checked that if $\beta(0) \in [0, 1]$, then $\beta(t) \in [0, 1], \forall t$.[3] Allowing X_1 and X_2 to be correlated, we can estimate the parameters of the joint covariance matrix as follows:

$$\begin{pmatrix} \sigma^2 & \rho\sigma\sigma_\beta \\ \rho\sigma_\beta\sigma & \sigma_\beta^2 \end{pmatrix} = \begin{pmatrix} \sigma_R^2 & \rho\sigma_R\sigma_A \\ \rho\sigma_A\sigma_R & \sigma_A^2 \end{pmatrix}$$

where

$$\begin{aligned} A(t) &= \log\left(\frac{1-\beta(t+h)}{\beta(t+h)} \frac{\beta(t)}{1-\beta(t)}\right) = \sigma_\beta X_2 \sqrt{h}, \ \forall t \\ R(t) &= r(t+h) - r(t) = K(t) + \sigma X_1 \sqrt{h} \end{aligned}$$

[3]This is one simple form of the recovery rate process which retains the feature that recovery rates must lie between 0 and 1, and also provides numerical tractability. This is without loss of generality as other modelling choices are not precluded in the model.

and $K(t)$ is a scalar constant.[4] Specification of the processes $[R(t), A(t)]$ permits the easy estimation of the parameters for the original processes $[r(t), \beta(t)]$. Normally, in discrete time, we would need to specify this bivariate process on a bivariate tree (4 branches emanating from each node).

However, since in our model the recovery rate enters as a *linear* scaling of cashflows (C), we can see that $E_{r,\beta}[C(\beta)] = E_r[C(E(\beta \mid r))]$. Therefore, we apply the law of iterated expectations and use only a univariate tree in $r(t)$ (only 2 branches emanating from each node), where we also obtain at each node the expected value of $\beta(t)$ given the value of $r(t)$ at that node. Therefore, we set up a binomial lattice in $r(t)$, with X_1 taking on values of (+1,-1) with equal probability. At each node in the tree, in addition to $r(t)$, we also compute the expected value of $\beta(t)$, which is given as

$$E[\beta(t) \mid r(t)] = E[\beta(t)] - \rho\frac{\sigma_\beta}{\sigma}[r(t) - E[r(t)]]$$

where

$$\begin{aligned} E[\beta(t)] &= \int_{-\infty}^{\infty}\left[1 + \frac{1-\beta(0)}{\beta(0)}\exp\left(\sigma_\beta \epsilon \sqrt{t}\right)\right]^{-1} \phi(\epsilon)\, d\epsilon \\ \epsilon &\sim N(0,1) \\ \phi(\epsilon) &= \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{\epsilon^2}{2}\right] \\ E[r(t)] &= f(0,t) + \sum_{j=0}^{\frac{t}{h}-1} \alpha(jh,t)h \end{aligned}$$

In this fashion we are able to set up the entire binomial tree in two state variables.[5]

[4] To see this note that

$$r(t+h) - r(t) = f(0,t+h) - f(0,t) + \sum_{j=0}^{\frac{t}{h}} \alpha(jh, t+h)h - \sum_{j=0}^{\frac{t}{h}-1} \alpha(jh, t+h)h + \sigma X_1\sqrt{h}.$$

[5] We are grateful to George Constantinides for suggesting the use of the law of iterated expectations in simplifying the exposition of the bivariate tree in interest rates and recovery rates as a simpler univariate tree.

3.2.3 *Default along the Sample Path:*

In their paper, Jarrow, Lando and Turnbull [13] assume that in the event of default prior to maturity, the claimholders would receive the recovery amount at *maturity* of the bond. This elegant assumption simplifies the risky term structure and the necessary computations quite considerably as the risky bond price is simply the state price times the expected cashflow in each state *at maturity.* However, it is tenable only under the assumption of constant recovery rates. Here, since we relax that assumption, we are not able to assume a single cashflow at maturity. Now, the time of default is important as it determines the recovery rate to be applied. A similar assumption to JLT was made by Longstaff and Schwartz [14]. However, under the assumption in this paper, the risky zero coupon bond will not generate a cashflow only at maturity, but may do so at several possible points along the sample path.

Let the current time be t. Define as before that $q_{iK}(mh)$ is the cumulative probability of default at time $(t+mh)$ Also note that $\beta(mh,n)$ is the recovery rate in the event of default at time $(t+mh)$ at the nth node. We introduce a new variable $q^0_{iK}(mh)$ which represents the one period probability of default over the period from $[t+(m-1)h]$ to $[t+mh]$. Using standard Markov chain analysis, it can be shown that the probability of default in the period indexed by m without default having occurred earlier (or the first passage time probability) is

$$[1-q_{iK}((m-1)h)]q^0_{iK}(mh)$$

The cumulative probability of default ending in the period indexed by m is given by:

$$q_{iK}(mh) = q_{iK}((m-1)h) + [1-q_{iK}((m-1)h)]q^0_{iK}(mh) \tag{7}$$

Therefore, the one period probability of default in the period indexed by m is

$$q^0_{iK} = \frac{q_{iK}(mh) - q_{iK}((m-1)h)}{1-q_{iK}((m-1)h)} \tag{8}$$

We now use these definitions to compute the expected cashflows over time from a zero coupon risky bond.

3.2.4 *Computing Sample Path Cashflows:*

We begin by examining the expected cashflows that will be generated by the risky zero coupon bond at each discrete time point in the sample path. Assume that the risky zero bond has a one dollar face value and this cashflow is due at time $T > t + h$. Since the bond can default prior to T, it is possible for the bond to generate cashflows prior to maturity. The expected cashflows at each state prior to maturity ($\tau < T$) will then be the recovery rate $[\beta(\tau)]$ times the first passage probability of early default at time τ. Denote the expected cashflow in the time period indexed by m as $C(m)$. For all time periods including maturity, the expected cashflows from the one dollar risky zero coupon bond with maturity T are as follows:

$$C(m) \;=\; \begin{array}{ll} q_{iK}(mh)\beta(mh,n), & m = \frac{t}{h}+1 \\ [1-q_{iK}((m-1)h)]q^0_{iK}(mh)\beta(mh,n), & \frac{t}{h}+1 < m < \frac{T}{h} \\ [1-q_{iK}((m-1)h)][1-q^0_{iK}(mh)(1-\beta(mh,n))], & m = \frac{T}{h} \end{array} \qquad (9)$$

The first line in the equation above is the expected cashflow when default occurs in the first time period from the valuation date. The third line provides the expected cashflow on the terminal date, given that no default has occured prior to maturity of the bond; and finally, the second line depicts expected cashflows in periods intermediate to the initial period and maturity.

3.2.5 *Computing Default Risk Premia [$\pi(.)$]:*

The expected value of all the risky cashflows discounted back along the term structure tree must be consistent with the prices in the initial term structure of interest rates and spreads. This leads to the following condition for the price of the risky zero-coupon bond promising to pay one dollar at maturity.

$$\begin{aligned} & \sum_{m=\frac{t}{h}+1}^{\frac{T}{h}} \left[E\left(\exp\left[-h \sum_{j=\frac{t}{h}}^{m-1} f(t,jh) \right] C(m) \right) \right] \\ = \;\; & \exp\left[-h \sum_{j=\frac{t}{h}}^{\frac{T}{h}-1} (f(t,jh) + s_i(jh)) \right] \times 1, \;\; \forall i \end{aligned} \qquad (10)$$

Equation (10) is simply the condition that, under the risk-neutral measure, the expected risky cashflows discounted at riskless rates must be equal to the value of expected riskless cashflows discounted at risky discount rates. Separating the last period's cashflows we can rewrite the above expression as

$$\begin{aligned}
& \sum_{m=\frac{t}{h}+1}^{\frac{T}{h}-1} \left[E\left(\exp\left[-h \sum_{j=\frac{t}{h}}^{m-1} f(t,jh) \right] C(m) \right) \right] \\
+ \quad & E\left(\exp\left[-h \sum_{j=\frac{t}{h}}^{\frac{T}{h}-1} f(t,jh) \right] [1 - q_{iK}(\frac{T}{h}-1)][1 - q_{iK}^0(\frac{T}{h})[1-\beta(\frac{T}{h},n)]] \right) \\
= \quad & \exp\left[-h \sum_{j=\frac{t}{h}}^{\frac{T}{h}-1} (f(t,jh) + s_i(jh)) \right], \quad \forall i
\end{aligned} \tag{11}$$

Define the state price at the nth node of the binomial tree at time t to be $w(t,n)$. The state prices are computed from the term structure model in Section 4.1. Incorporating this definition, the equation above is as follows:

$$\begin{aligned}
& \sum_{m=\frac{t}{h}+1}^{\frac{T}{h}-1} \left[\sum_{n=1}^{m} w(mh,n)[1 - q_{iK}((m-1)h)] q_{iK}^0(mh)\beta(mh,n) \right] \\
+ \quad & \sum_{n=1}^{m} w(\frac{T}{h},n)[1 - q_{iK}(\frac{T}{h}-1)][1 - q_{iK}^0(\frac{T}{h})[1-\beta(\frac{T}{h},n)]] \\
= \quad & \exp\left[-h \sum_{j=\frac{t}{h}}^{\frac{T}{h}-1} (f(t,jh) + s_i(jh)) \right], \quad \forall i
\end{aligned} \tag{12}$$

Re-arranging and solving for $q_{iK}^0(mh)$ gives us the following equation:

$$q_{iK}^0(\frac{T}{h}) = \frac{A - B + C}{D} \tag{13}$$

where

$$\begin{aligned}
A &= \sum_{m=\frac{t}{h}+1}^{\frac{T}{h}-1} \left[\sum_{n=1}^{m} w(mh,n)[1 - q_{iK}((m-1)h)] q_{iK}^0(mh)\beta(mh,n) \right] \\
B &= \exp\left[-h \sum_{j=\frac{t}{h}}^{\frac{T}{h}-1} (f(t,jh) + s_i(jh)) \right] \\
C &= \sum_{n=1}^{m} w(\frac{T}{h},n)[1 - q_{iK}(\frac{T}{h}-1)]
\end{aligned}$$

$$D = \sum_{n=1}^{m} w(\frac{T}{h}, n)[1 - q_{iK}(\frac{T}{h} - 1)][1 - \beta(\frac{T}{h}, n)]], \ \forall i \tag{14}$$

From the above expressions, it is clear that $D > 0$, because the state prices $w(t,n) > 0$ by definition, and $q_{iK}(.), \beta(.) \in [0,1]$. For similar reasons, $A > 0, B > 0, C > 0$. We can also see that $A - B + C > 0$ since it is equal to $q^0_{iK}(.)\, D > 0$. The fact that $q_{iK} > 0$ follows from equations (7) and (8). From equation (8) we can see that since $q_{iK}(mh) - q_{iK}((m-1)h) > 0$ (the cumulative probability of default must be increasing), the default probabilities will always lie in the range [0,1] as required. Finally, having computed $q^0_{iK}(\frac{T}{h})$ we obtain the cumulative probability of default using equation (7):

$$q_{iK}(\frac{T}{h}) = q_{iK}(\frac{T}{h} - 1) + [1 - q_{iK}(\frac{T}{h} - 1)]q^0_{iK}(\frac{T}{h})$$

Once we obtain the risk-neutral probabilities of default $q_{iK}(.)$, the risk premia can be computed using the following equation:

$$\pi(i)(jh) = \frac{q_{iK}(jh)}{p_{iK}}, \ \forall i \tag{15}$$

They can be then be used in conjunction with the no-arbitrage term structure to compute the prices of several kinds of risky debt.

Whereas in the current model structure, we use a statistical transition matrix, which is assumed to be time homogenous, no such restriction is necessary. It is possible to estimate the P matrix for several time periods. Then, instead of employing the one period P matrix raised to the required power, we can use the time-specific matrix, in equation (15).

Finally, the model may be enhanced to place more structure on the Markov chain for credit ratings. In the current version of the model, the initial Markov chain (P) is obtained empirically. By positing a functional relationship between the probabilities in the transition probability matrix, it may be possible to express them as a function of a few state variables. This will allow fitting the Markov chain to a cross-section of risky debt prices. Whereas this approach will be more useful remains an empirical question.

3.3 Interpreting Default Spreads

Our model provides for risky term structures, stochastic credit ratings, and stochastic spreads. Default spreads are a function of the probability of default as well as the amount lost when default occurs. In other words, stochastic forward spreads are a function of the risky ratings (embodied in the Markov chain) and stochastic recovery rates (implemented on the bivariate tree).

The *cumulative* risk-neutral probability of default for any rating i after a period T is given by the q_{iK}th element of the $Q(T)$ matrix. The risk-neutral probability of default in forward period T in rating i is then given by $\frac{q_{iK}(T)-q_{iK}(T-h)}{1-q_{iK}(T-h)}$. At each node (T,l,n) on the tree we also know the spot rate $r(T,l,n)$ and the recovery rate at the two attached nodes in the next period $\beta(T+h,l,n)$. The one-period forward spread $s_i(T,l,n)$ at node (T,l,n) on the tree is then computed from the following equation:

$$\exp[-hs_i(T,l,n)] = E\left[1 - \frac{q_{ik}(T+h)-q_{iK}(T)}{1-q_{iK}(T)}(1-\beta(T+h,l,n))\right], \quad \forall i$$

which gives the solution as follows:

$$s_i(T,l,n) = -\frac{1}{h}\log\left(E\left[1 - \frac{q_{ik}(T+h)-q_{iK}(T)}{1-q_{iK}(T)}(1-\beta(T+h,l,n))\right]\right), \quad \forall i \tag{16}$$

This equation simply expresses the forward spread as a function of the stochastic recovery rates in the future time period. These simple computations on the bivariate tree allow us to express the entire spread structure as well. The expression for spreads also clearly defines the joint impact of default probabilities and recovery rates on the determination of spreads. Ay increase in the risk-neutral probability of default will increase spreads, and any increase in the recovery rate will decrease spreads. Moreover, since we know that $\beta(.), q_{iK}(.) \in [0,1]$ and also $q_{iK}(T+h) > q_{iK}(T)$, it is clear that the spreads $s_i(.)$ will always also be greater than zero, which is desirable.

Our model is now complete. We are able to represent on a bivariate tree, stochastic interest rates, recovery rates and spreads, as well as embed changing credit ratings in the model. Our model provides risk-neutral valuation, free of arbitrage for any credit-sensitive instrument whose payoffs can be expressed as a function of any of these stochastic variables.

As summary, a schematic of the model is presented in Figure 1. The figure depicts the various elements of risky debt prices which must be consistent with each other. The three key elements of the model are the HJM term structure, the default process provided by the Markov chain for credit ratings (P), and the stochastic recovery rate process. The term structure and recovery rates are correlated with coefficient ρ. The interaction of these three components in a risk-neutral pricing environment must be consistent with the observed term structure of credit spreads. The risk-neutral drifts on the term structure dimension are obtained as desscribed in Section 4.1. The term structure and recovery rates processes are combined on a bivariate tree upon which we impose the risk-nuetral default process Markov chain (Q). The statistical transition probabilities P are converted into risk-neutral probabilities Q so as to be consistent with observed spreads using a risk premium adjustment $\pi(.)$ and the stochastic recovery rate process. This entire structural scheme is then used for the pricing of risky debt.

4 Applications of the Model

This model can be used to value a broad variety of debt contracts, including plain vanilla debt, callable debt, floating-rate notes, bonds with credit-risk puts, credit-sensitive notes, spread-adjusted notes, credit-risk derivatives, and swaps. In this section, we use the model to price some of these instruments, and discuss its use in pricing one credit-contingent contract, credit-sensitive notes.

4.1 Categorizing Risky Debt

It is useful to categorize debt instruments on the basis of the risks borne by investors. Consistent with the model developed, we identify exposure to shifts in the riskless term structure, changes in the probability of default (change in firm rating), and changes in the recovery rate (credit spread conditional on rating). If we then list a variety of debt instruments, we can see how they have varying sensitivities to these different risks.

Plain-vanilla (fixed rate) risky debt is subject to all three risks as the promised coupons of the corporate coupon bond are fixed and do not adjust with any of the risk factors. Typical floating rate notes (FRNs), where the coupons vary with the risk-free rate, protect the holder from changes in the risk- free rate, but leave the holder exposed both to changes in credit rating and changes in the appropriate spread conditional on credit rating. Instruments that try to eliminate investor's exposures to all of these risks include auction- rate notes, whose coupons are reset each period to those prevailing in the market in an attempt to trade at par on each reset date. In theory, the auction rate process is designed to provide the holder with adjustments in the promised coupons so that changes in any of the three risks are perfectly offset.

The other instruments shown in the table below provide investors with protection against some but not all of these risks. Credit-sensitive notes, which are discussed in detail in the following section, pay off cashflows which are based on a prespecified schedule of fixed coupons contingent on the credit rating level at each reset date in the future. While they provide holders with protection against changes in rating, they do not provide them with any protection against changes in the spreads conditional on those ratings. Spread adjusted notes (SPANs) are instruments where the underlying riskless rate is fixed, but the spread over this rate is adjusted to prevailing spreads in the market, based on a predetermined credit rating. Because the spread over the fixed rate changes for both rating changes and changes in the appropriate spread given any rating level, these instruments are subject only to interest rate risk.

The features of these instruments are summarized in the following table:

	Price Sensitivity to		
Security	Interest Rate Risk	Recovery Rates (Spread Risk)	Probability of Default (Default Risk)
Plain Vanilla Debt	Yes	Yes	Yes
Swaps	Yes	Yes	Yes
FRN	No	Yes	Yes
Auction Notes	No	No	No
CSN	Yes	Yes	No
SPAN	Yes	No	No

4.2 Applications of the Pricing Model - Pricing Stylized Securities

To provide the reader with the sensitivities of values to changing model parameters, we use the model to price a variety of instruments listed in the preceeding table, with the results provided in Figures 2 and 3. In all cases, we assume that debt has a maturity of five years. In Figure 2, we plot the prices of risky debt of various types against the level of the short rate. The credit rating assumed for the debt is A. Pure risky debt is assumed to carry a coupon of 9%. The FRN is assumed to pay the short-term rate plus 90 basis points. The SPAN is based on a floor riskless rate of 8% plus the stochastic spread. And finally, the CSN has a coupon schedule for each of the 7 non-default ratings as follows: (8,8.1,8.5,9,10,10.5).

As can be seen only the FRN shows little sensitivity to the change in the term structure. This is because the coupon adjusts with the changing short rate. The value of the FRN is not totally immune to changes in interest rate levels, because some effect of changing interest rates enters the valuation of the FRN through the correlation between the spreads and the term structure.

In Figure 3, prices are plotted against changes in the initial recovery rate, which is the proxy for the level of spreads in the economy. Here, the instrument with low sensitivity is the SPAN as its cashflows adjust with changing spreads.

The pricing of swaps where the counterparties to the swap have differential risk ratings is possible in our model because we include spreads for all possible credit ratings. We assume that in the event of default by any party to the swap, the other party is still obliged to make good on his payments, and that the payments due are net (not gross) payments. In practice, there are several possible settlement scenarios based on default, and our aim here is not to analyze each one in detail, but merely to provide an indicative example. Let us assume a swap where we contract to pay fixed, and receive floating payments at the riskless rate plus 60 basis points. We also assume that our current rating is A (i=3) and that of the payer of the floating rate is AAA (i=1). We compute swap values over a range of interest rates and recovery rates. We also assume that the fixed rate can be one of three values: 9.0,9.5 and 10.0 percent per annum. The maturity of the swap is 5 years. Figure 4 presents the value of the three swaps for varying interest rates, and Figure 5 provides the same analysis for different recovery rates. In Figure 4 it is easily seen that as interest rates increase the swap which is net negative NPV becomes positive in value, since the magnitude of the floating rate payments increases. Of course the swap value is declining in the fixed rate. In Figure 5 we notice that there is a mild decrease in value of the swap to the payer of the fixed rate when the recovery rate increases. This is because as the recovery rate increases, it benefits the party with the better credit rating, as his expected losses are reduced. The effect here on prices is minimal however, because of the small difference between the creditworthiness of the parties to the swap. In Figure 6, we assume the same scenarios as in Figure 5, but amend the credit rating of the fixed rate payer is BB ($i = 5$). The slope of the lines is higher in comparison to those in Figure 5 as the difference in credit ratings is larger.

4.3 Analyzing Credit Spreads

In Section 4.3, we depicted an analytic expression (equation 16) for forward credit spreads. This equation established the influence of both default probabilities and recovery rates on the magnitude of spreads. In this section, we examine the impact of changes in these two factors on spreads.

The results are presented in Table 7, and show the spreads for a range of recovery rates, and default probabilities. The results clearly demonstrate that credit risk is sensitive to both the factors, and that spreads do vary a lot even when the probability of default is held constant. Therefore, allowing recovery rates to be stochastic provides the extra degree of volatility we observe empirically when spreads seem to vary even when the rating of a bond has not changed. Our model provides a simple way to capture this feature in the realm of practical application.

4.4 Application of the Model - Pricing Enron's Credit-Sensitive Note

Credit-sensitive notes (CSNs) are innovative debt instruments which would be difficult to price using many of the extant models. A CSN is a debt instrument whose payoff is contractually linked to the credit rating of the issuer. A typical CSN would be issued with a fixed rate schedule, such that the issuer would promise to pay predetermined coupons on the CSN based on current credit rating. Were the issuer's credit rating not to change, these fixed interest payments would not change. However, if the issuer's credit rating deteriorated or improved, its coupon payments would increase or decrease, respectively, according to the preset schedule. Therefore, the payoff on a CSN varies only with credit rating changes and not with interest rates or spreads. In this sense, they make for interesting study, as pure credit-contingent instruments.

CSNs enjoyed brief notoriety beginning in May 1989 when Enron Corporation publicly offered the first CSN.[6] In the period 1989-90, there were several issues amounting to a cumulative issue size of about $2.3 billion. See Table 6.

Comparing a CSN to a traditional FRN reveals the salient differences. On a traditional FRN, the issuer pays a fixed spread over a Treasury yield or LIBOR, and the promised coupon adjusts for changes in interest rates. The FRN investor is protected against riskless interest rate changes, but bears all of the risk of changes in the issuer's credit rating or changes in the spread for each ratings category. In contrast, with a CSN, changes in interest rates do not affect the coupon paid; coupons are adjusted only due to changes in the issuer's credit rating. For example, on the Enron issue, the initial coupon was set at 9.5%, and were Enron's credit rating not to change, this coupon would remain unchanged. However, if Enron's rating were to fall from Baa to Ba , the coupon would rise to 12%. Conversely, were its credit standing to improve to A, the coupon would fall to 9.4%. The holder of the CSN bears not only interest rate risk (as the owner of a fixed-rate obligation), but also "spread risk," i.e., the risk that the change in yield for a change in rating matches the market rate for that credit spread. Therefore, it is easy to see that pricing the CSN requires analysis wherein we cannot assume that interest rates, credit ratings or credit spreads are constant.

[6] While the Enron issue was the first public offering, a variant of the CSN structure is reported to have been used extensively in revolving credit facilities by banks. In these transactions, interest rates are keyed off of the levels of accounting variable such as interest coverage and leverage.

S&P Rating	AAA	AA	A	BBB	BB+	BB	BB-	<= B
Promised Enron Coupon (%)	9.2	9.3	9.4	9.5	12.0	12.5	13.0	14.0
Yield on Corporate Bonds (%)	9.33	9.62	10.10	10.44	–	11.88	–	12.91

CSNs were created as a reaction to increasing investor concern about event risk in the late 1980s. The junk bond era was nearing its end, and credit spreads reached historically high levels. Moreover, the differential in spreads between investment grade and junk debt had also widened. Therefore, the CSN provided a natural hedge to investors against credit rating downgrades, by imposing the cost of rating declines on the issuer in a precommitted agreement. The cost to the issuer would be offset by potential gains from improvements in credit ratings. Unlike the FRN, by establishing an aggressive rate schedule on issue of the CSN, the issuer was able to credibly signal its view of better prospects. Therefore, the CSN could provide a mechanism for firms to credibly signal their prospects of credit improvements. In theory, they might also lead to lower agency costs of debt. Any risk-shifting by management in an effort to transfer value from the debtholders to the equityholders will be reflected in the credit rating, and the concurrent increase in the coupon transmits these transfers back to the debt holders. The issuer of the CSN bears downgrade risk, in exchange for the potential upside gain, signaling opportunities, and lower financing costs.

Based on its apparent alternatives, it is unclear whether the CSN was attractive to Enron, at least on the basis of its price. The initial coupon Enron would pay were its credit not to change was lower than the rate it would have paid on a fixed-rate obligation, consistent with other researchers' findings.[7] If Enron's credit improved, it would pay slightly less than better-rated fixed-coupon issuers would have paid. However, if its credit worsened, it committed itself to paying substantially more than lower-rated fixed-rate issuers would have promised to pay.

Inputs to the model to value the Enron CSN were obtained from several sources. The initial forward rate curves for May and June 1989 was obtained from the McCulloch and Kwon [16] database. The volatility of the forward rate was estimated using a time series of

[7] Ogden and Moon [19] find that issuers of reset notes and credit sensitive notes pay initial coupons below those associated with fixed rate bonds of similar characteristics. They do not examine the full schedule of payments.

data from this same database. Recovery rate data was obtained from the published tables in Altman's study [1]. The correlation (17%) between the term structure and recovery rates was computed with the same data as described so far. The term structure of spreads for June 1989 was obtained from Standard and Poors [24]. Finally, the Markov probabilities of going from one rating level to another were taken from the ratings transition matrices published in the Standard & Poor's Credit Review of January 25, 1993. The ratings matrix also includes non-rated issues. These were apportioned to the lowest three rating classes equally, assuming that the large proportion of unrated firms are drawn from sub-investment grade issues.[8]

The Enron CSN issue was offered on May 31, 1989 for par (settlement on June 15). Using data as of this date, the model produces a value of 100.4, reasonably close to the offering price quoted at the beginning of of the selling period. By the end of June 1989, the model suggests that the price should have risen to 104.0, based on changes in interest rates over that month. According to Moody's Bond Record, the closing price for June 1989 was 103.91, including accrued interest. Thus, the model appears to be able to produce prices that closely resemble traded prices for a complicated credit-sensitive instrument.

The Enron CSN is substantially more valuable than risky debt, as evidenced by the fact that at the end of June, the bond's actual and model price were approximately 104, but BBB-rated debt with a coupon of 9.5% would have sold for 107.9. The skewed nature of the rate schedule offered by Enron can explain the reason why CSN investors would demand materially lower returns than investors in a similarly-rated 9.5% fixed-rate note. In the event of worsening ratings, the CSN coupon would rise substantially. But in the event of ratings deterioration, it would decline slowly. Effectively, Enron provided substantial downside protection to the investors without demanding much 'give-back' in the event of improvements in credit ratings, and thus, the embedded rating option was quite valuable.

[8] The application of the model to pricing actual issues of exotic risky debt is simple and feasible. The model was implemented on a Pentium 60 Mhz PC and generated the CSN price in about 1 minute. Programming of the model was undertaken using Mathematica [27] which is run through an interpreter, and run times would be much faster if a compiled language such as C++ were used instead.

4.4.1 Fitting Parameters to Match Market Prices

Using historical data to estimate parameters, we priced the Enron CSN and obtained prices which were different from market values. Alternatively, it is possible to imply the parameters from market prices. As an example, let us assume that we obtained and fit the term structure parameters to the existing prices of riskless bonds. To imply the default probabilities and recovery rate levels, we can use the price of the Enron CSN.

Figure 8 demonstrates that a wide range of possible values of default probabilities and recovery rates would match the prices of Enron's credit sensitive note. In the figure, we plot the price of the CSN for varying levels of the recovery rate (β) and what we denote the Q-scale. The Q-scale is a multiplier attached to the P matrix governing the 'speed' of default. This is carried out as follows. If for example, we need the 2 period probability of default, we raise the one period P matrix to the power of 2. If we wish to increase the 2 period rate of default, we can also do this by raising the P matrix to a power greater than 2. On the other hand, if we wish to reduce the rate of default, we raise this matrix to a power less than 2. If we wish to leave the probability of default unchanged, the Q-scale is equal to unity.

We carried out this exercise and computed values depicted in Figure 8. Several pairs of recovery rate values and Q-scale values cover a wide range of possible CSN prices, and demonstrates the ease with which this model may be used to fit parameters to observed risky debt prices.

5 Conclusion

This paper highlights three sources of risk in pricing credit-sensitive debt: interest-rate risk, default risk, and recovery-rate (spread) risk. The innovation of the paper is the recognition that spreads and recovery rates are stochastic, and may be correlated with the term structure. This simple enhancement has several modeling benefits: (i) it provides greater variability in spreads, in line with that observed in practice, (ii) it enables a stochastic decomposition of credit spreads into likelihood of default and recovery on default, (iii) it allows spreads to vary even when the firm's rating class does not change, (iv) it injects correlation between spreads and the term structure of interest rates, (v) it enables firm and security-specific features of spreads to be accommodated, and (vi) it enables the pricing of a wide range of spread-based exotic debt and options.

As a practical contribution, the paper uses a discrete-time model that is easily implementable using ordinarily observable inputs. The model is capable of pricing a wide range of credit-sensitive exotica, and is also useful to price a broad range of risky debt contracts. We demonstrate this by using the model to price an actual credit-sensitive note using data, and find that the model produces values quite close to those observed in practice.

References

[1] Altman, E. "Revisiting the High-Yield Bond Market," *Financial Management*, 1992, v21(2), 78-92.

[2] Amin, K. "On the Computation of Continuous Time Option Prices Using Discrete Approximations," *Journal of Financial and Quantitative Analysis*, 1991, v26(4), 477-496.

[3] Amin, K. and J. Bodurtha. "Discrete Time Valuation of American Options with Stochastic Interest Rates," *Review of Financial Studies*, 1994, forthcoming.

[4] Bhattacharya, S. and S.P. Mason. "Risky Debt, Jump Processes and Safety Covenants," *Journal of Financial Economics*, 1981, v9(3), 281-307.

[5] Black, F. and J.C. Cox. "Valuing Corporate Securities: Some Effects of Bond Indenture Provisions," *Journal of Finance*, 1976, v31(2), 351-367.

[6] Das, Sanjiv. "Credit Risk Derivatives," *Journal of Derivatives*, 1995, v2(3), pg 7-23.

[7] Duffie, D., and K. Singleton. "Econometric Modelling of Term Structures of Defaultable Bonds," 1994, Working Paper, Stanford University.

[8] Duffee, G. R. "The Variation of Default Risk with Treasury Yields, " 1995, Unpublished manuscript.

[9] Harrison, J.M. and D.M. Kreps. "Martingales and Arbitrage in Multiperiod Security Markets," *Journal of Economic Theory*, v20, 381-408.

[10] Heath, D., R.A. Jarrow, and A.J. Morton. "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation," *Econometrica*, 1992, v60(1), 77-105.

[11] Ho, T. & S. Lee. "Term Structure Movements and Pricing Interest Rate Contingent Claims," *Journal of Finance*, 1986, v41, pp 1011-1029.

[12] Jarrow, R. and S.M. Turnbull. "Pricing Options on Financial Securities Subject to Credit Risk," *Journal of Finance*, 1995, 53-86.

[13] Jarrow, R., D. Lando, and S.M. Turnbull. "A Markov Model for the Term Structure of Credit Risk Spreads," 1994, Working Paper, Cornell University.

[14] Longstaff, F.A., and E.S. Schwartz. "A Simple Approach to Valuing Risky Fixed and Floating Rate Debt," 1994, Working Paper, UCLA.

[15] Lucas, D.G., "The Effectiveness of Downgrade Provisions in Reducing Counterparty Credit Risk," *Journal of Fixed Income*, 1995, 32-41.

[16] McCulloch, H., and Heon-Chul Kwon. "U.S. Term Structure Data 1947-91," 1993, Ohio State University, WP 93-6.

[17] Merton, R.C. "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates," *Journal of Finance*, 1974, v29, 449-470.

[18] Nielsen, T.N., J. Saa-Requejo, and P. Santa-Clara. "Default Risk and Interest Rate Risk: The Term Structure of Default Spreads," 1993, Working Paper, INSEAD.

[19] Ogden, J. and T. Moon. "An Analysis of Reset Notes and Rating Sensitive Notes" *Journal of Financial Engineering*, 1993, v2,2, 175-194.

[20] Shimko, D. "The Valuation of Risky Discrete Cashflows in Continuous Time," *Journal of Finance*, 1989, v44(5), 1373-1384.

[21] Shimko, D., N. Tejima, and D.R. Van Deventer. "The Pricing of Risky Debt when Interest Rates are Stochastic," *Journal of Fixed Income*, 1993, Sept, pp 58-65.

[22] Smith, C., and J. Warner. "On Financial Contracting: An Analysis of Bond Covenants," *Journal of Financial Economics*, 1979, v7(2), 117-162.

[23] Spahr, R.W., M.A. Sunderman, and C. Amalu. "Corporate Bond Insurance: Feasibility and Insurer Risk Assessment," *Journal of Risk and Insurance*, 1991, v58(3), 418-437.

[24] Standard and Poor's *Standard and Poor's Corporate Finance Criteria*, 1994.

[25] Tufano, P. "Do Fools Rush In? Rewards to Buying and Selling the Newest Financial Products, " 1993, Unpublished manuscript.

[26] Vasicek, O. “An Equilibrium Characterization of the Term Structure”, *Journal of Financial Economics*, 1977, v5, 177-188.

[27] Wolfram. S. “Mathematica: A System for doing Mathematics by Computer”, Addison-Wesley.

Table 1: **Descriptive Statistics**

This table presents descriptive statistics for yields on Treasuries, yields on corporate bonds and the spreads between corporate yields and the 30 year constant-maturity Treasury bond. Data covers the period 1976-1991.

	Descriptive Statistics				
Variable	Mean	Median	Std-dev	Min	Max
Treasuries					
CMT-2	9.26	8.67	2.62	4.36	16.46
CMT-5	9.56	8.85	2.35	5.84	15.93
CMT-10	9.77	9.03	2.16	6.84	15.32
CMT-30	9.81	9.01	1.97	7.27	14.68
Corporates					
AAA	10.50	9.62	2.00	7.92	15.50
AA	10.80	9.83	2.12	8.12	16.00
A	11.20	10.10	2.27	8.37	16.50
BAA	11.80	10.70	2.35	8.80	17.20
Spreads					
AAA	0.66	0.67	0.31	-0.20	1.61
AA	1.03	0.98	0.38	0.39	2.01
A	1.46	1.34	0.58	0.54	3.27
BAA	1.96	1.88	0.64	0.92	3.76

Table 2: **Correlation of Corporate and Riskless Yields**

This table presents the correlation of corporate yields and yields of constant-maturity Treasuries. The data spans the period 1976-1991.

	Corporate Yields			
Rates	AAA	AA	A	BAA
Treasuries				
CMT-2	0.92	0.92	0.90	0.90
CMT-5	0.97	0.97	0.95	0.95
CMT-10	0.99	0.98	0.97	0.97
CMT-30	0.99	0.98	0.97	0.97

Table 3: **Correlation of Spreads with Yields and Spreads**
This table provides the correlation coefficient of the spreads with the other variables in our analysis. The data is monthly and covers the period 1976-1991.

	Credit Spreads over CMT-30s			
Rates	AAA	AA	A	BAA
Treasuries				
CMT-2	0.01	0.25	0.35	0.44
CMT-5	0.03	0.28	0.40	0.48
CMT-10	0.04	0.30	0.42	0.50
CMT-30	0.02	0.29	0.42	0.49
Spreads				
AAA	1.00	0.89	0.80	0.77
AA		1.00	0.95	0.93
A			1.00	0.97
BAA				1.00

Table 4: **Correlations of Changes in Credit Spreads with Changes in Other Variables**
In this table we present the correlation of changes in credit spreads with changes in the Treasury and corporate yields. The data spans the period 1976-1991, and is monthly.

	Changes in Credit Spreads			
Changes in	AAA	AA	A	BAA
Treasuries				
CMT-2	-0.16	-0.31	-0.38	-0.42
CMT-5	-0.25	-0.38	-0.46	-0.49
CMT-10	-0.32	-0.48	-0.55	-0.56
CMT-30	-0.39	-0.52	-0.59	-0.60
Spreads				
AAA	1.00	0.79	0.70	0.67
AA		1.00	0.91	0.85
A			1.00	0.91
BAA				1.00

Table 5: **Correlations between credit spreads and recovery rates**
This table presents the correlation between recovery rates and default spreads. The data is annual for the period 1985-1991. The data was obtained from Altman [1].

	Spreads over CMT-30s			
Debt Class	AAA	AA	A	BAA
Recovery Rate				
Secured	-0.40	-0.30	-0.30	-0.20
Senior	-0.20	-0.20	-0.10	-0.20
Senior Sub.	-0.30	-0.50	-0.50	-0.60
Subordinated	-0.40	-0.70	-0.80	-0.80

Table 6: **Fixed Rate Credit-sensitive Note Issues**
The table summarizes public offerings of CSNs.

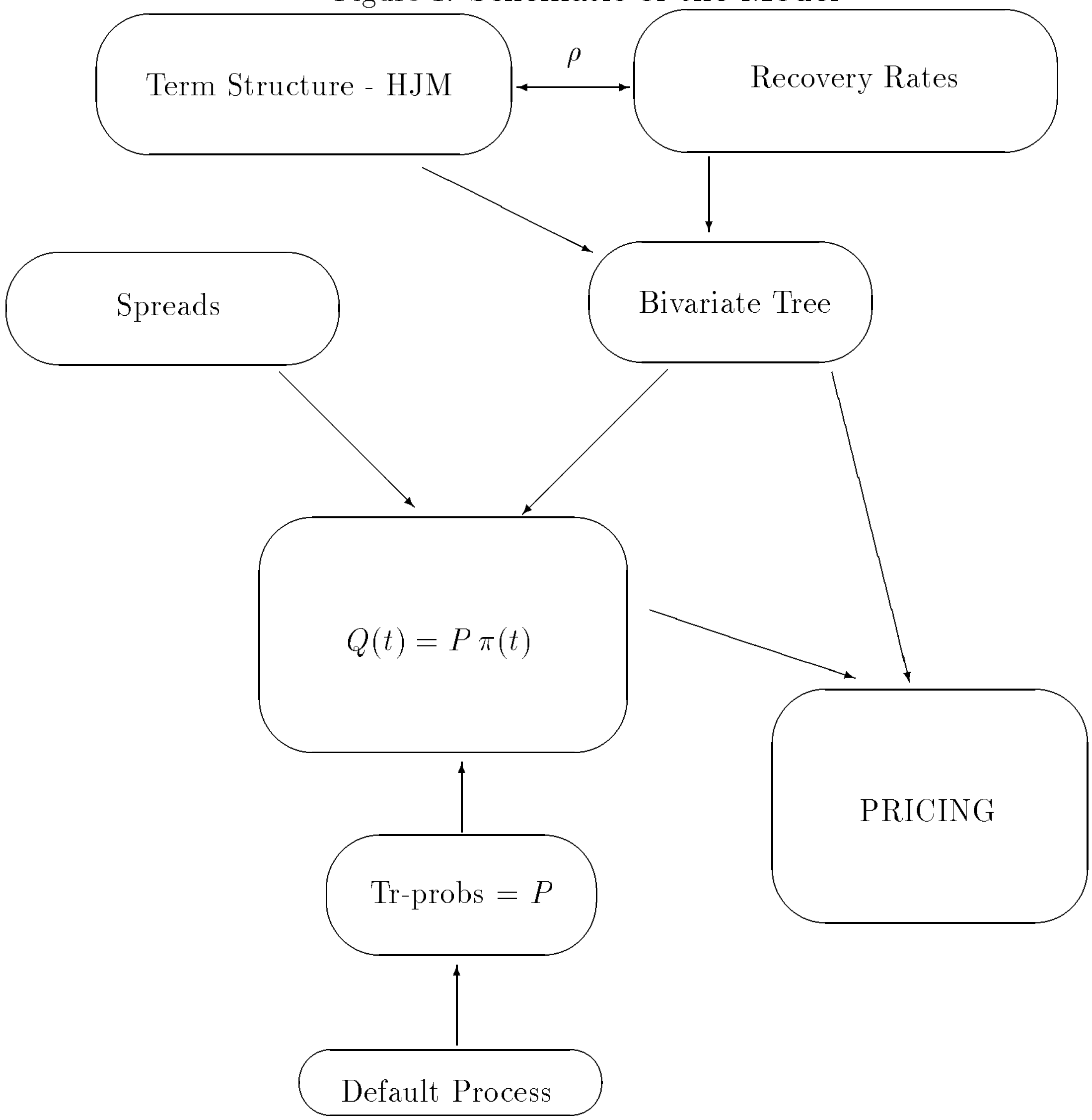
Date	Issuer	S&P Rating at issue	Maturity in years	Size ($ MN)
05-31-89	Enron Corp.	BBB-	12	100
12-06-89	Potlatch Corp.	A-	20	100
04-27-90	Auburn Hills Trust (Chrysler guarantee)	BBB-	30	1000
05-21-90	Unisys Corp.	BBB	7	300
06-05-90	Morton International	AA-	30	200
06-06-90	Georgia Pacific	BB+	7	300
06-06-90	Georgia Pacific	BB+	12	300
Total				2300

Table 7: **Spread Sensitivity to Changes in the Probability of Default and Changes in Recovery Rates**

This table presents an analysis of how forward credit spreads (for a 10 year maturity) change when the two spread components, (a) default probabilities and (b) recovery rates change. The spreads are computed using equation (16) . We use a range of recovery rates (0.20, 0.34, 0.50), and a range of default probabilities, at 75%, 100% and 125% of the empirical risk-neutral probabilities,which are presented below, for cumulative default at 9.5 and 10 years. Therefore the default probabilities are scaled by the probability multipliers 0.75, 1.0 and 1.25 respectively in the table. Sensitivity to both factors exists, as is quite apparent from the table, and this is presented for each rating level.

	Rating →	AAA	AA	A	BBB	BB	B	C
Cum.Prob of default	10yr	0.1283	0.1565	0.1998	0.261	0.3648	0.5035	0.7667
	9.5yr	0.1215	0.1482	0.1894	0.2475	0.3465	0.4795	0.7358
					Spreads			
Prob Multiple	**Recov rt**	AAA	AA	A	BBB	BB	B	C
0.75	0.20	0.0039	0.0049	0.0063	0.0087	0.0130	0.0198	0.0367
0.75	0.34	0.0032	0.0040	0.0052	0.0072	0.0107	0.0163	0.0302
0.75	0.50	0.0024	0.0030	0.0040	0.0054	0.0081	0.0123	0.0228
1.00	0.20	0.0054	0.0068	0.0090	0.0126	0.0197	0.0326	0.0853
1.00	0.34	0.0044	0.0056	0.0074	0.0103	0.0162	0.0268	0.0698
1.00	0.50	0.0034	0.0042	0.0056	0.0078	0.0122	0.0203	0.0523
1.25	0.20	0.0070	0.0089	0.0119	0.0171	0.0285	0.0537	0.4223
1.25	0.34	0.0058	0.0073	0.0098	0.0141	0.0234	0.0440	0.3320
1.25	0.50	0.0044	0.0055	0.0074	0.0107	0.0177	0.0331	0.2391

Figure 1: **Schematic of the Model**



INPUTS	INTERMEDIATE OUTPUTS	END RESULTS
The Term Structure Model: • Forward Rate Curve [f(.)] • Volatility of Forward Rates [σ, λ]	• Risk Neutral Drifts [α] • Term Structure Tree	
Default Model: • Spreads [s_i] • Transition probability matrix [P] • Recovery Rate level [β] • Recovery Rate volatility [σ_β] • Correlation of recovery rates and interest rates [ρ]	• Default Risk Premia [π] • Sample Path Cashflows [C(m)] • Risk Neutral Default Probabilities [Q,q] $\Downarrow$	
Stylized Risky Debt Instrument: Timing and amount of cashflows Other specific structural details	$\Rightarrow$ COMPUTE $\Rightarrow$	Price of contract

Figure 2: **Risky Debt Prices for Varying Interest Rates**
The plot here consists of risky debt prices of pure debt, Floating Rate Notes, Spread Adjusted Notes and Credit-sensitive Notes. It is apparent that the FRN is insensitive to interest rates. All debt is assumed to have a maturity of 5 years, and an A rating. Pure risky debt is assumed to carry a coupon of 9%. The FRN is assumed to pay the riskless rate plus 90 basis points. The SPAN is based on a floor riskless rate of 8% plus the stochastic spread. And finally, the CSN has a coupon schedule for each of the 7 non-default ratings as follows: (8,8.1,8.5,9,10,10.5).

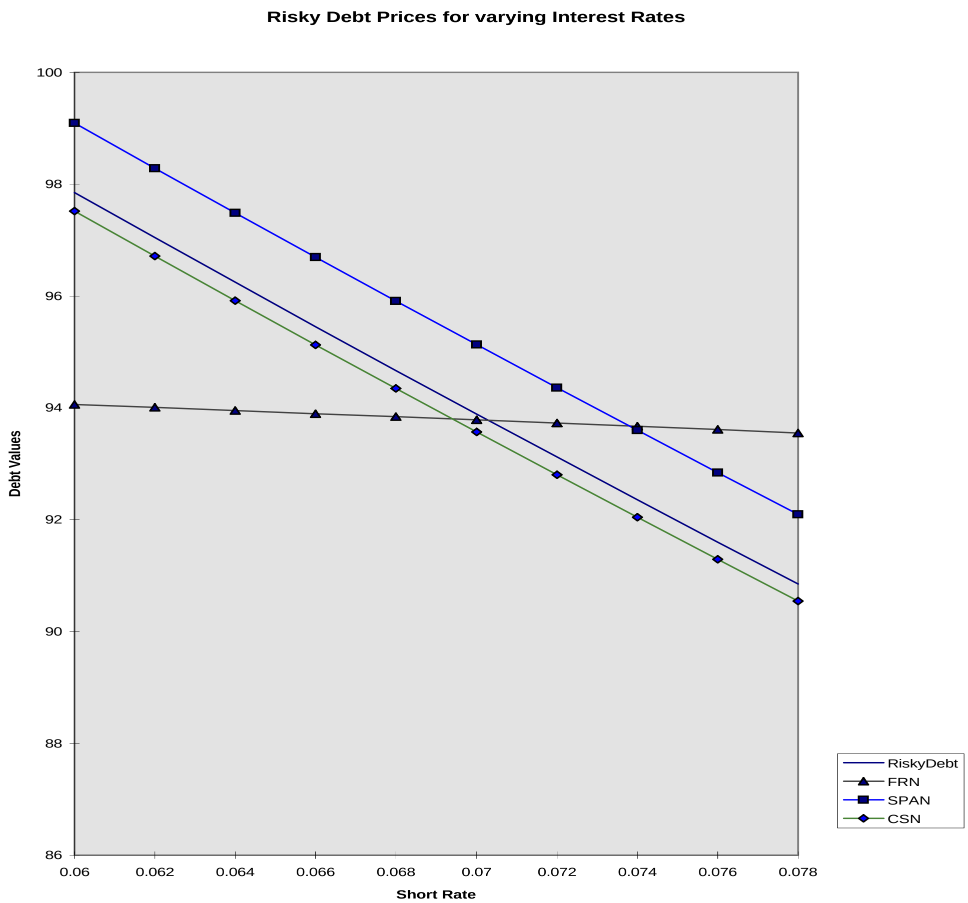


Figure 3: **Risky Debt Prices for Varying Recovery Rates**
The plot here consists of risky debt prices of pure debt, Floating Rate Notes, Spread Adjusted Notes and Credit-sensitive Notes. It is apparent that the SPAN is insensitive to recovery rates, or spreads. All debt is assumed to have a maturity of 5 years, and an A rating. Pure risky debt is assumed to carry a coupon of 9%. The FRN is assumed to pay the riskless rate plus 90 basis points. The SPAN is based on a floor riskless rate of 8% plus the stochastic spread. And finally, the CSN has a coupon schedule for each of the 7 non-default ratings as follows: (8,8.1,8.5,9,10,10.5).

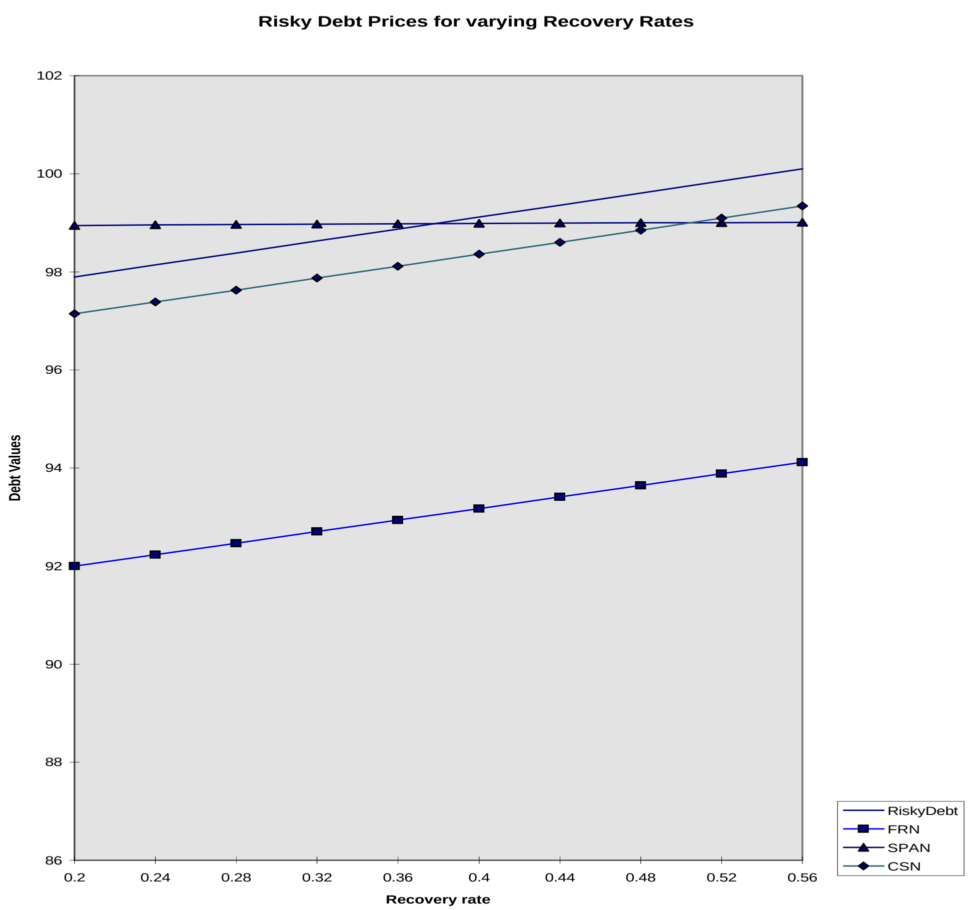


Figure 4: **Swap Prices for Varying Interest Rates**
The plot here consists of swap prices at various interest rates for three swaps where the fixed leg is the pay side at 9.0, 9.5 and 10.0 percent. The swap maturity is 5 years, and coupons are semi-annual. The receive side is floating at the riskless rate plus 60 basis points. The credit rating of the floating rate payer is AAA and that of the fixed rate payer is A. The initial recovery rate is 40%, and the initial spot rate is 6%. The values presented are the swap NPVs from the point of view of the payer of the fixed rate.

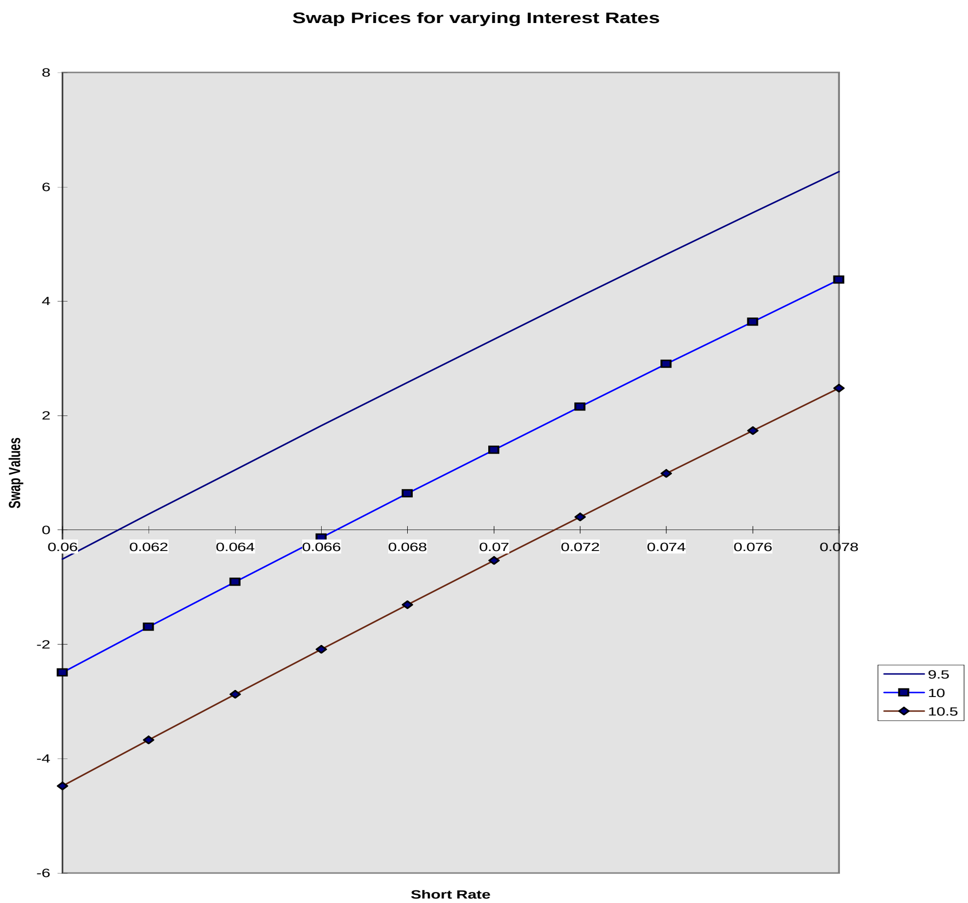


Figure 5: **Swap Prices for Varying Recovery Rates**
The plot here consists of swap prices at various recovery rates for three swaps where the fixed leg is the pay side at 9.0, 9.5 and 10.0 percent. The swap maturity is 5 years, and coupons are semi-annual. The receive side is floating at the riskless rate plus 60 basis points. The credit rating of the floating rate payer is AAA and that of the fixed rate payer is A. The initial recovery rate is 40%, and the initial spot rate is 6%. The value presented are the swap NPVs from the point of view of the payer of the fixed rate.

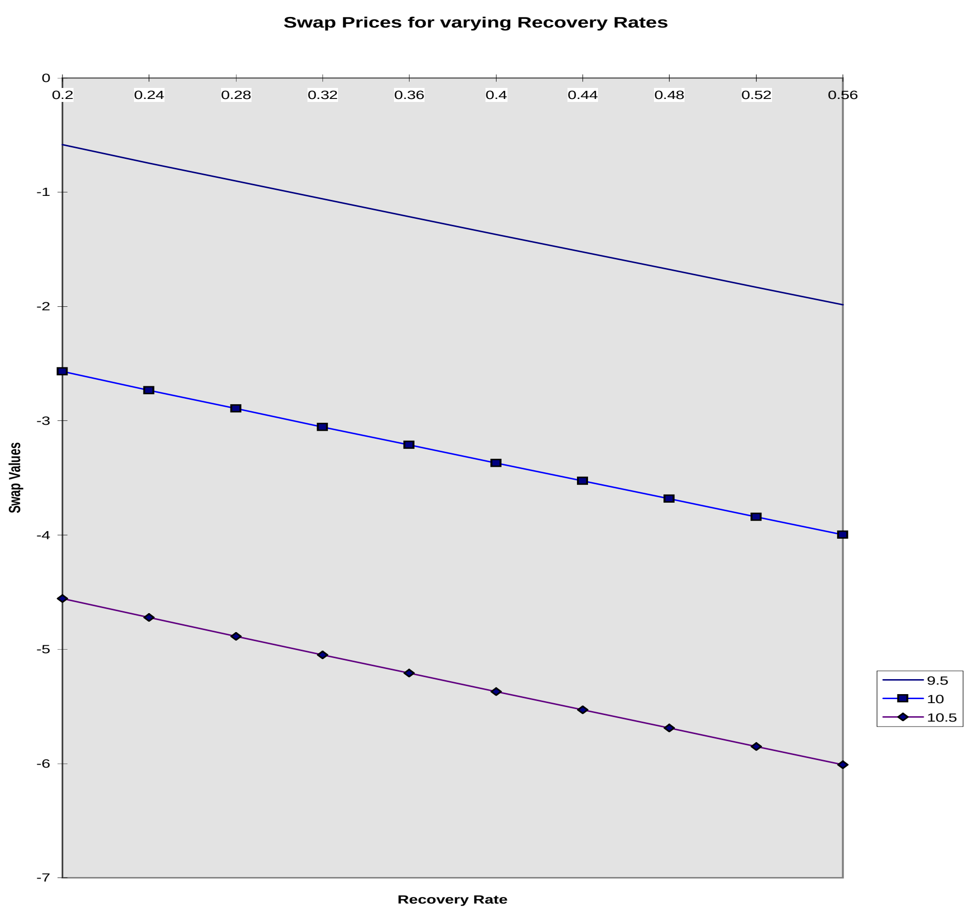


Figure 6: **Swap Prices for Varying Recovery Rates**
The plot here consists of swap prices at various recovery rates for three swaps where the fixed leg is the pay side at 9.0, 9.5 and 10.0 percent. The swap maturity is 5 years, and coupons are semi-annual. The receive side is floating at the riskless rate plus 60 basis points. The credit rating of the floating rate payer is AAA and that of the fixed rate payer is BB. The initial recovery rate is 40%, and the initial spot rate is 6%. The value presented are the swap NPVs from the point of view of the payer of the fixed rate.

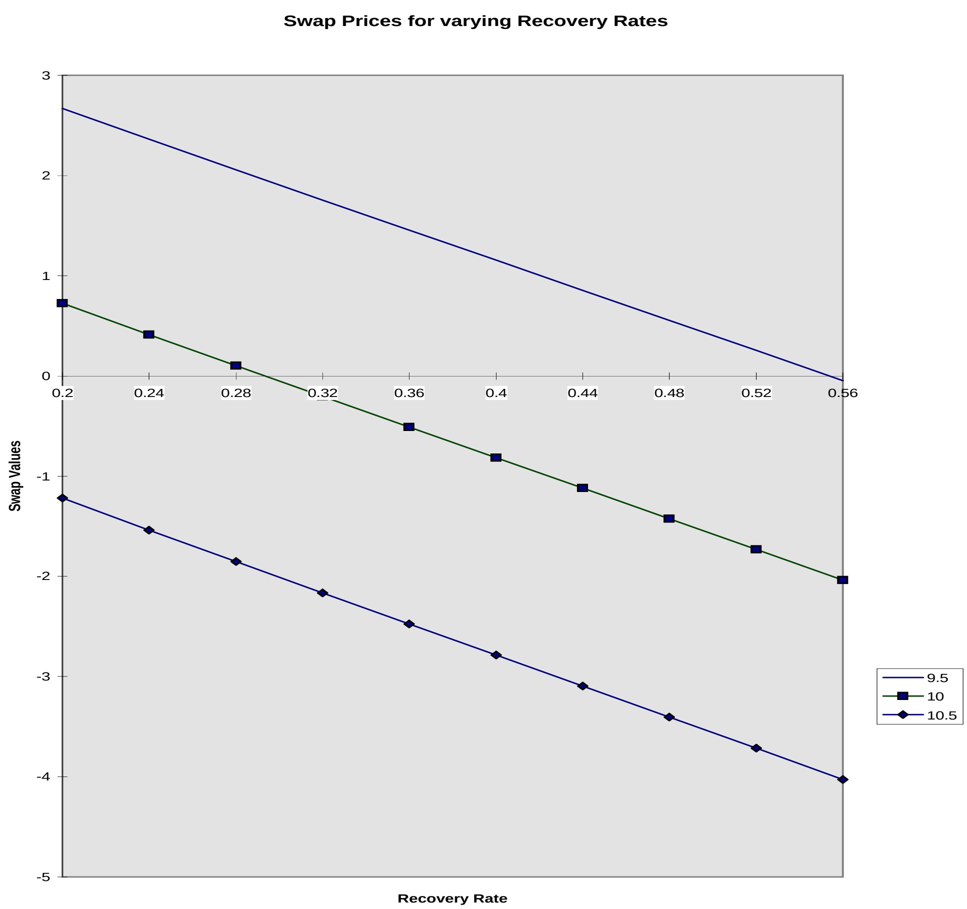


Figure 7: **Enron CSN Prices for varying Recovery Rates** Presented in this graph are CSN prices for varying recovery rates, for the mid-June 1989 issue of Enron Corp. Enron issued these notes paying semi-annual interest off a coupon schedule depending on the current rating of the company. The June 1989 forward rate curve from McCulloch and Kwon was used. Interest rate volatility was 0.0156, recovery rate volatility 0.0063, correlation between the term structure and recovery rates was 0.17, and the mean spreads for each credit rating were (0.0082, 0.0101, 0.0131, 0.0175, 0.0255, 0.0375, 0.0675, 0.1). The current rating of the company is BBB. The graph also shows prices of corresponding risky and riskless coupon debt. The coupon is 9.5%.

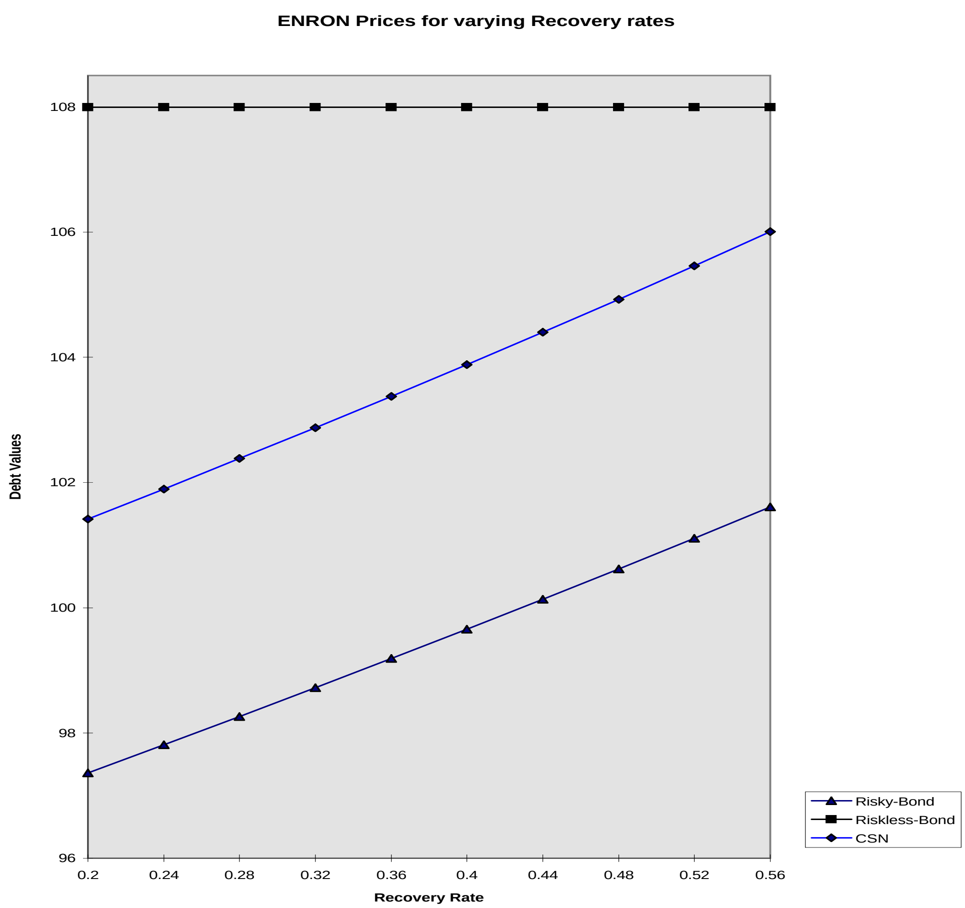


Figure 8: **Enron CSN Prices**
Presented in this graph are CSN prices for varying recovery rates and default probabilities, for the mid-June 1989 issue of Enron Corp. Enron issued these notes paying semi-annual interest off a coupon schedule depending on the current rating of the company. The June 1989 forward rate curve from McCulloch and Kwon was used. Interest rate volatility was 0.0156, recovery rate volatility 0.0063, correlation between the term structure and recovery rates was 0.17, and the mean spreads for each credit rating were (0.0082, 0.0101, 0.0131, 0.0175, 0.0255, 0.0375, 0.0675, 0.1). The current rating of the company is BBB. The graph axes are as follows: Beta depicts the variation in the recovery rate, and Q-scale stands for the multiplier of the rate of bankruptcy. The *Q-scale* is a convenient way of modifying the default probabilities. At a Q-scale value of 1, the P matrix is unaffected. When Q-scale is greater than 1, it means that we accelerate the speed at which bankruptcy occurs by a factor of the Q-scale. Similarly, when the Q-scale is lower than 1, it means that the probability of default reduces.