The Role of Options in Goals-Based Wealth Management

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Abstract

We develop a facile methodology using dynamic programming for goals-based wealth management over long horizons where rebalancing uses the standard securities and also derivative securities. A kernel density estimation approach is developed to accommodate any number of derivative assets, solving a high dimensional problem with fast computation. The approach easily accommodates skewed and fat-tailed distributions. Portfolio performance is much better with the use of options, especially for investors with aggressive goals.

Keywords: Options; goal-based investing; dynamic programming; kernel estimators
JEL Codes: G11; G13

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1 Introduction

Dynamic portfolio management has had a long history since the work of Merton (1969); Merton (1971), extending static optimization ideas in Markowitz (1952). Long-horizon wealth management has usually been undertaken using equities and bonds, but not derivatives, though there are several arguments made for the use of these securities, such as diversification, hedging, speculation, enhancing leverage, downside protection, reaching for goals, efficient rebalancing, etc., as noted in Hoogendoorn et al. (2017). Since the crisis of 2008, diversification across asset classes has declined, triggering the need for alternate approaches to improve the risk-return trade-off in portfolios through the use of options and volatility derivatives (Guobuzaite and Martellini (2012); Jones (2014)) because positions in volatility help hedge market risk (Bakshi and Kapadia (2003); Arsic (2005)).

It has been argued that structured products such as options are unsuitable for retail investors as they are too complex to be understood and pose risks that may be unacceptable (McCann and Luo (2006)), or these products are used inappropriately with little benefit (Branger and Breuer (2008)). Even institutional asset managers have not reaped the benefits of derivatives use in their portfolios, see Fong et al. (2005); Beber and Perignon (2013). However, in recent times, the wealth management industry has begun focusing on goals, and it is also becoming clear that achieving goals is likely to become easier when options are used. In this paper, we implement an enhanced goals-based wealth management algorithm (GBWM, see Shefrin and Statman (2000); Nevins (2004); Chhabra (2005); Brunel (2015)) that includes taking positions in call and put options on the index. This extends existing GBWM algorithms [Browne (1995); Browne (1997); Browne (1999a); Browne (1999b); Browne (2000); Das et al. (2010); Wang et al. (2011); Deguest et al. (2015); Das et al. (2018); Das et al. (2020)] that only include stocks, bonds, and indexes but not derivative securities. One can envisage that the use of options will make it easier to manage a portfolio over time to reach specified goals. This paper assesses how much the performance of GBWM models can be improved through the use of options in addition to standard securities. This paper also develops an interesting new approach to dynamic programming of the wealth management strategy using dimension reduction via kernel density estimators.

Options are especially useful in reaching goals, as we will show subsequently in this paper. The results in this paper complement a history of work on the construction of options portfolios where the mean-variance paradigm is inapplicable, see for example early work by Liu and Pan (2003), and recent work by Faias and Santa-Clara (2017) who maximize expected utility (accounting for all moments of returns) instead of the Sharpe ratio (which trades off mean versus variance of returns). In our modeling, utility maximization is replaced by maximizing the probability of reaching the investor's goals. This is analogous to imposing VaR constraints as in Kleindorfer and Li (2005). Our approach applies whether or not the conditions for two-fund separation (Cass and Stiglitz (1970)) apply, especially since, with options, return distributions are not compatible with mean-variance assumptions.

This paper makes methodological advances and also offers analyses showing how simple options may be used to improve dynamic wealth management. The contributions are as follows: First, standard mean-variance methods in static models are woefully inadequate for structuring dynamic goal-based portfolios with options, as the dynamics of geometric Brownian motion do not capture higher-order moments of returns, and do not capture properly the complexities of multivariate return distributions that are involved. In standard dynamic portfolio problems, there is only a single stochastic variable, i.e., portfolio return, composed of a weighted sum of asset returns, usually assumed to be Gaussian. With more asset classes, optimal portfolios may need to be chosen using multivariate Gaussian distributions, which poses no issues because the formulations of much of the computation involved are available in closed form. However, when derivatives are included, multivariate distributions are no longer Gaussian, nor are they amenable to implementation via copula functions. The conditional distribution of portfolio wealth needed for dynamic programming is a univariate composition over highly skewed, non-Gaussian multivariate distributions. We also need to compute these conditional distributions exceedingly fast in order to be able to implement a practically useful dynamic model. Section 2.3.4 shows how this is done using a combination of simulation and fast kernel density estimation. This approach is extensible to projecting any high dimensional distribution of asset and option returns on the univariate wealth transition probability function.

Second, since the approach taken in this paper is a numerical one, it extends the results in Liu and Pan (2003) by enabling additional features that may not yield closed-form solutions. These are features such as different objective functions that are different from utility maximization, including infusions and withdrawals in the portfolios, closing out and rolling options positions over time, and permitting any distribution of asset returns, especially non-Gaussian ones.

Third, in the setting of goals-based optimization, we show that call options are effective and put options are not. There are two reasons for this. One is that puts are negative expected return investments and unless they are absolutely necessary to meet goals, they are mathematically inoptimal instruments. Two, since goals are usually high thresholds and not floors on portfolio value, calls are the natural choice.

Fourth, we see that as goals become more aggressive, calls are used more, and the difference in performance of a wealth management strategy with and without the use of options becomes more marked. Investors with higher goals are better off when using options. For example, for an investor with an initial wealth of \$100, and a 10-year goal of reaching \$250, who can invest up to 30% of the portfolio at any time in calls, the probability of reaching her goal increases from 69% without the use of options to 86% when call options are used.

Fifth, we also assess whether a mostly options strategy may be sufficient and find this not to be the case. This is simply because using index options only is less effective than using a range of portfolios from the efficient frontier. Of course, using a large range of possible options on many assets may improve comparative performance.

Sixth, we consider how the use of options helps when we do not restrict the use of options to only 30% of the portfolio, allowing, when optimal to increase option use to 90% of the portfolio. The improvement in outcomes is material, especially for aggressive goals, such as the one mentioned earlier. In that case, more option use pushes up the probability of reaching the goal from 86% to 96%, suggesting that the use of options results in a first-order improvement in portfolio outcomes, complementing the results of Guidolin (2013).

Finally, we also explore the effect of fat-tailed distributions by changing the mean-variance portfolios from being based on Gaussian distributions to fatter-tailed ones (a t distribution with 5 degrees of freedom). This helps proxy for the fat-tails induced by jumps and stochastic volatility. Interestingly, we find that the probability of reaching goals reduces by a very small amount ($\sim 1\%$). However, mean returns on the portfolio increase but are offset by increases in return standard deviation, which is only to be expected as the tails of the distributions are substantially fatter.

The rest of the paper proceeds as follows. Section 2 describes the dynamic programming algorithm and the novel procedure for accommodating derivatives in wealth management through the use of kernel density estimation. Section 3 offers several analyses and insights related to the results above. Concluding discussion is in Section 4.

2 Dynamic Programming

This paper undertakes standard dynamic programming as in papers like Deguest et al. (2015); Das et al. (2020). The approach assumes standard stochastic processes for the evolution of wealth in a goals-based portfolio and an objective function defined in the ensuing subsections.

2.1 Objective Function

The GBWM objective function stipulates the maximization of the probability of reaching a threshold level of wealth H at time horizon T, i.e.,

$$\max_{w(t), t<T} Pr[W(T) > H] \tag{1}$$

where a sequence of portfolios $w(t), t = 0, h, 2h, ..., T-h$, at periodic interval h, are chosen to dynamically achieve the highest probability of exceeding threshold H.[1] This is a standard optimal control problem.

2.2 Portfolios in the Choice Set

For the examples in this paper, we ensure that all portfolios used in the dynamic solution lie on the efficient frontier. These portfolios are solved for using the seminal solution in Markowitz (1952). This solution provides all possible portfolios that are mean-variance optimal over a single period. At each time t, we choose any one efficient portfolio $w(t) \in \mathcal{R}^n$, comprised of n possible choice assets. This portfolio is characterized by a mean return $\mu = w^\top M$ and variance of return $\sigma^2 = w^\top \Sigma w$, where $M \in \mathcal{R}^n$ is a vector of expected returns on the n assets in the portfolio, and $\Sigma \in \mathcal{R}^{n \times n}$ is the covariance of returns. We require that $\sum_{j=1}^n w_j = 1$, i.e., all the money is fully allocated to the portfolio assets.

The mean-variance optimization problem yields the minimized portfolio return variance σ^2 for a chosen level of portfolio expected return μ, subject to the full wealth allocation constraint.

[1]Note that $w(t)$ is a vector of portfolio weights and $W(t)$ is the scalar value of the portfolio through time.

The solution to this problem is available from Markowitz (1952). For different chosen μ we get a collection of optimal portfolio pairs (μ, σ), known as the "efficient frontier", from which we may choose to compile a sequence of optimal portfolios, $w(t)$, each of which map onto a mean and standard deviation of return $[\mu(t), \sigma(t)]$. In other words, we solve the dynamic programming problem of goals-based wealth management by optimally rebalancing to one of a set of efficient portfolios at every discrete time point in the model. This set of candidate efficient portfolios may be independently determined and may even be chosen using criteria that are different from Markowitz mean-variance optimization.

In addition to mean-variance portfolios, we also allow the investor to buy call and put options on any asset. In the examples in the paper, we restrict ourselves to at-the-money options on a stock index and therefore the benefits from trading options that we evidence in our analyses may be understated.

2.3 Wealth Transition Functions

2.3.1 Transitions without options

Without loss of generality, we define the stochastic change in wealth in the portfolio to be governed by geometric Brownian motion, i.e.,

$$W(t+h) = W(t) \exp\left[\left(\mu(t) - \frac{1}{2}\sigma(t)^2\right) h + \sigma\sqrt{h} \cdot Z(t)\right], \quad Z(t) \sim N(0,1) \tag{2}$$

This is standard, but not required, any other stochastic process can be substituted here. The transition probability function is directly derived from equation (2).

$$Pr[W(t+h)|W(t)] = \phi(x) \tag{3}$$

where

$$x = \frac{\ln(W(t+h)/W(t)) - (\mu(t) - \frac{1}{2}\sigma(t)^2)h}{\sigma\sqrt{h}} \tag{4}$$

where $\phi(\cdot)$ is the standard normal probability function.

2.3.2 Grid points

We establish a discrete set of grid points in wealth levels to define the two-dimensional state space $[W(t), t]$ for our problem. These points should cover a wide range of values of wealth that are likely to be reached from initial wealth $W(0)$. Our scheme establishes the maximal range of wealth as follows, accounting for a 4σ move, up or down, in log wealth over time, using a high level of standard deviation, denoted σ_{max}:

$$W(t+h) \in [\exp^{\ln(W(0)) - 4\sigma_{max}\sqrt{T}}, \exp^{\ln(W(0)) + 4\sigma_{max}\sqrt{T}}] \tag{5}$$

This range is discretized on a grid of $(m+1)$ values $[W_0(T), W_1(T), ..., W_m(T)]$, with an odd number of points over m intervals of width k in logspace, i.e.,

$$\ln W_i(T) - \ln W_{i-1}(T) = k, \quad \forall i = 1, 2, ..., m \tag{6}$$

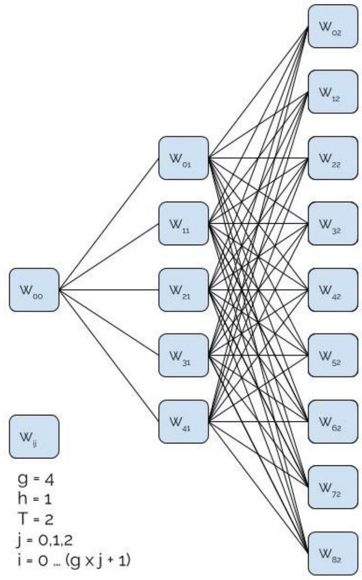

Figure 1: Sample grid with the following parameter values: time interval $h = 1$ year; multiplier $g = 4$; horizon $T = 2$ years; time points $j = 0, 1, 2$; and the number of nodes at each time point, $i = g \cdot j + 1$. Notice that there will be an additional g nodes for each additional period added to the grid.

The number of time intervals is T/h. We define an even numbered multiplier g, such that the number of grid points at the end of interval j will be $(g \cdot j + 1)$. Note that at time T, $m = g \cdot (T/h)$, and the number of grid points at time T is $(m + 1)$. Figure 1 shows a sample grid for just two periods.

We will solve the dynamic program on this grid using the Bellman equation, detailed in Section 2.4, by implementing standard backward recursion, computing the value function starting from time $t = T$ backwards to time $t = 0$. To take expectations for computing the value function, we need to compute transition probabilities between portfolio wealth values $W(t)$ and $W(t+1)$, which depend on the stochastic process above and for options. With options, this is more complicated than in Section 2.3.1. We turn to describing this aspect of the dynamic program next.

2.3.3 Transitions with options

A fraction of the portfolio wealth may be invested in call and put options. This will change the transition probability function, without necessitating a change in the grid itself. Define as $C(t)$ the value of an at-the-money call option on the stock index $I(t)$, and $P(t)$ is the corresponding put value. Assume that the chosen horizon for these options is always the time per period, i.e., interval h. We can use any option pricing model to get these prices, but for simplicity we assume that the Black and Scholes (1973); Merton (1973) model is deployed. In this case the index follows a geometric Brownian motion, which is

$$I(t+h) = I(t) \exp\left[\left(\mu_I - \frac{1}{2}\sigma_I^2\right) h + \sigma_I \sqrt{h} \cdot Z_I\right] \tag{7}$$

where μ_I is the mean return on the index and σ_I is the standard deviation. The correlation between Brownian motions Z and Z_I is denoted as ρ. The value of an at-the-money call option on the index with maturity h is

$$C(t) = I(t)[N(d_1) - e^{-rh}N(d_2)] \equiv I(t) \cdot X_c \tag{8}$$

and for puts the price is

$$P(t) = I(t)[e^{-rh}N(-d_2) - N(-d_1)] \equiv I(t) \cdot X_p \tag{9}$$

where the risk free rate is denoted r and

$$\begin{aligned} d_1 &= \frac{1}{\sigma_I}\left(r + \frac{1}{2}\sigma_I^2\right)\sqrt{h} \qquad (10) \\ d_2 &= \frac{1}{\sigma_I}\left(r - \frac{1}{2}\sigma_I^2\right)\sqrt{h} \qquad (11) \end{aligned}$$

We assume that there is a fixed proportion α_c of the portfolio that may be invested in calls, and α_p in puts. If no investment is made in any option, then the situation defaults to the transitions described in equation (3).

The number of calls and puts invested in is as follows, i.e., the wealth invested in options divided by the price of the option:

$$\begin{aligned} n_c(t) &= \frac{\alpha_c(t) \cdot W(t)}{C(t)} = \frac{\alpha_c(t) \cdot W(t)}{I(t) \cdot X_c} \qquad (12) \\ n_p(t) &= \frac{\alpha_p(t) \cdot W(t)}{P(t)} = \frac{\alpha_p(t) \cdot W(t)}{I(t) \cdot X_p} \qquad (13) \end{aligned}$$

The net wealth left for investment in non-derivatives after investment in the options is

$$W'(t) = W(t)[1 - \alpha_c - \alpha_p] \tag{14}$$

where α_c, α_p could also be zero. This wealth will evolve under equation (2) with chosen mean and standard deviation $[\mu(t), \sigma(t)]$.

Given the value of the stock index $I(t)$ at time t, the payoff of at-the-money options at time $t+h$ will be $\max[0, I(t+h) - I(t)]$ for calls and $\max[0, I(t) - I(t+h)]$ for puts. Therefore, total wealth will evolve as follows:

$$\begin{aligned} W(t+h) &= W'(t)\exp\left[\left(\mu(t) - \frac{1}{2}\sigma(t)^2\right)h + \sigma(t)\sqrt{h} \cdot Z(t)\right] \\ &\quad + n_c(t)\max[0, I(t+h) - I(t)] + n_p \max[0, I(t) - I(t+h)] \end{aligned} \tag{15}$$

The transition probability density function is now dependent on joint outcomes of the wealth invested in options, which depends on the evolution of $I(t)$, and that not invested in options, which depends on the evolution of $W'(t)$. The correlation ρ between the index and wealth also matters.

We elaborate equation (15) as follows:

$$\begin{aligned} W(t+h) &= W(t)[1-\alpha_c-\alpha_p]\exp\left[\left(\mu-\frac{1}{2}\sigma^2\right)h+\sigma\sqrt{h}\cdot Z\right] \\ &\quad +\frac{\alpha_c(t)\cdot W(t)}{I(t)\cdot X_c}\max[0, I(t+h)-I(t)] \\ &\quad +\frac{\alpha_p(t)\cdot W(t)}{I(t)\cdot X_p}\max[0, I(t)-I(t+h)] \end{aligned} \tag{16}$$

which can then be written as follows, noting that the right-hand side of the equation is independent of wealth levels:

$$\begin{aligned} \frac{W(t+h)}{W(t)} &= [1-\alpha_c-\alpha_p]\exp\left[\left(\mu-\frac{1}{2}\sigma^2\right)h+\sigma\sqrt{h}\cdot Z\right] \\ &\quad +\frac{\alpha_c(t)}{X_c}\max[0, I(t+h)/I(t)-1] \\ &\quad +\frac{\alpha_p(t)}{X_p}\max[0, 1-I(t+h)/I(t)] \end{aligned} \tag{17}$$

Using equations (7), (10), and (11), we further obtain:

$$\begin{aligned} \frac{W(t+h)}{W(t)} &= [1-\alpha_c-\alpha_p]\exp\left[\left(\mu-\frac{1}{2}\sigma^2\right)h+\sigma\sqrt{h}\cdot Z\right] \\ &+ \frac{\alpha_c(t)}{[N(d_1)-e^{-rh}N(d_2)]}\max\left\{0, \exp\left[\left(\mu_I-\frac{1}{2}\sigma_I^2\right)h+\sigma_I\sqrt{h}\cdot Z_I\right]-1\right\} \\ &+ \frac{\alpha_p(t)}{[e^{-rh}N(-d_2)-N(-d_1)]}\max\left\{0, 1-\exp\left[\left(\mu_I-\frac{1}{2}\sigma_I^2\right)h+\sigma_I\sqrt{h}\cdot Z_I\right]\right\} \end{aligned} \tag{18}$$

We are therefore able to write the transition $W(t)$ to $W(t+h)$ as a ratio, $R(t)=\frac{W(t+h)}{W(t)}$, which is a function only of the primitives of the problem, i.e., the 8 parameters

$$\{\alpha_c, \alpha_p, \mu, \sigma, \mu_I, \sigma_I, h, r\}$$

and two correlated random variables $\{Z, Z_I\}$, which have correlation ρ. As we can see, $R(t+h)$, which is 1 plus the return, is independent of the level of wealth $W(t)$. This means we can compute the probability density (pdf) for returns, $\ln(R)$ for a given set of parameters only once and re-use it repeatedly. In other words $\ln(R(t))$ does not depend on t or $W(t)$ and may be written simply as $\ln(R)$.

How many sets of pdfs will we need? Suppose we have 15 possible (μ, σ) efficient portfolios and choose the proportion in calls to be either of $\{0, \alpha_c\}$, and the proportion in puts to be $\{0, \alpha_p\}$. Then, all told, we pre-compute $60 = 15\times 2\times 2$ sets of pdfs and store these to provide all possible transition probability functions.

2.3.4 Transition probabilities with options using kernel density estimators

In order to generate the probability density function (pdf) for R we need to use the joint distribution for $\{Z, Z_I\}$. The simplest way to do this is to generate a large number M of correlated pairs of values from this joint distribution, and then use these values in equation (18) to generate M values of $\ln(R)$, all of which are equally likely. We then fit a kernel density function to the data on $\ln(R)$ to get the pdf. The procedure would be as follows:

1. Generate M correlated random variable pairs $\{Z, Z_I\}$ using the following scheme (say, $M = 5,000$):
   - Generate an independent standard random normal variate pair $(e_1, e_2) \sim N(0, 1)$.
   - Set $Z = e_1$.
   - Set $Z_I = \rho \cdot e_1 + \sqrt{1 - \rho^2} \cdot e_2$.
   - Repeat M times and store the final results.
2. Given a configuration of the parameters, generate M values of $\ln(R)$ using equation (18).
3. Fit a kernel density estimator to the M values of $\ln R$ to get the pdf. Denote this as $f(\ln R)$. We fit a Gaussian kernel density estimator (KDE) to the returns using standard Python functions, i.e., the fast `gaussian_kde` function, based on O'Brien et al. (2016).
4. Repeat this for all 60 parameter configurations.

Given a level of wealth $W(t)$, and future levels of wealth on grid points $[W_0(t+h), ..., W_m(t+h)]$, we get ratios of wealth by dividing the latter by the former, to get $[R_0, R_1, ..., R_m]$. Because these are discrete points, we convert the transition probability pdf into a discrete probability vector where

$$Pr(\ln R_i) = \frac{f(\ln R_i)}{\sum_{i=0}^{m} f(\ln R_i)} \geq 0 \tag{19}$$

which assures that $\sum_{i=0}^{m} Pr(\ln R_i) = 1$.

Sample program code to implement this scheme in Python is shown in Figure 2.

```python
from scipy.stats import norm
from scipy.stats import gaussian_kde as KDE

def Rpdf(alpha_c, alpha_p, mu, sig, muI, sigI, h, r, rho):
    e1 = randn(10000)
    e2 = randn(10000)
    z = e1
    zi = rho*e1 + sqrt(1-rho*rho)*e2
    d_1 = (r+0.5*sigI**2)*sqrt(h)/sigI
    d_2 = (r-0.5*sigI**2)*sqrt(h)/sigI
    R = (1 - alpha_c - alpha_p)*exp((mu-0.5*sig*sig)*h+sig*sqrt(h)*z) + \
        alpha_c/(norm.cdf(d_1)-exp(-r*h)*norm.cdf(d_2)) * maximum(0,exp((muI-0.5*sigI**2)*h+sigI*sqrt(h)*zi)-1) + \
        alpha_p/(exp(-r*h)*norm.cdf(-d_2)-norm.cdf(-d_1)) * maximum(0,1-exp((muI-0.5*sigI**2)*h+sigI*sqrt(h)*zi))
    R = log(R)
    kernel = KDE(R)
    return kernel
```

Figure 2: Python code to generate the transition probability kernel. In practice, especially for the implementation of the dynamic programming algorithm, the basic kernel density estimation (KDE) function runs somewhat slow and we use a fast KDE algorithm available in Python as well, O'Brien et al. (2016).

We implemented the code to generate four density functions for cases with and without options and these are displayed in Figure 3.

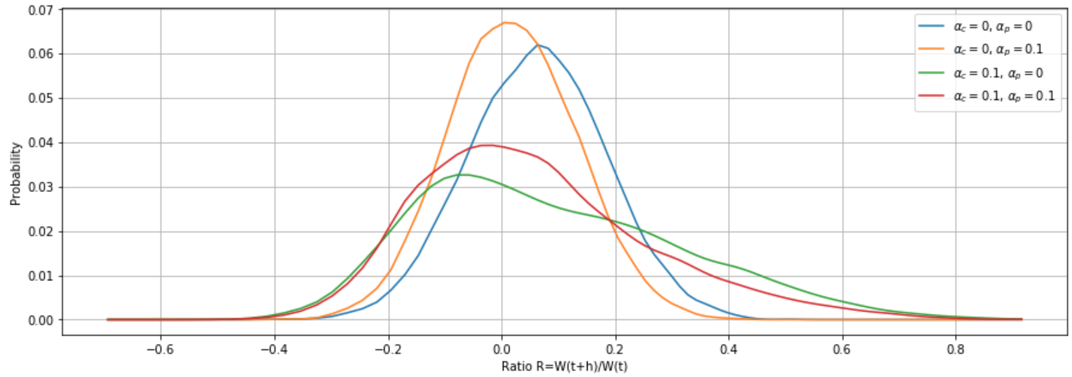

Figure 3: Probability density functions (pdfs) for the distribution of $R = W(t+h)/W(t)$ when calls and puts are used. The following parameters are used to generate the density functions: the fraction of the portfolio in calls (puts) is α_c (α_p); returns: $\mu = 0.07$, $\sigma = 0.12$, $\mu_I = 0.08$, $\sigma_I = 0.18$, $\rho = 0.6$; time interval is $h = 1$ and the risk free rate is $r = 0.01$. All four cases are shown with the base case being no options, and the other cases use one of or both calls and puts.

2.4 Optimization using backward recursion

Our approach is to determine a dynamic trading strategy to maximize the probability of exceeding the goal threshold H, as specified in equation (1). This is a standard dynamic programming problem that calls for backward recursion on a two-dimensional grid in wealth $W(t)$ and time t, constructed as per Section 2.3.2. For ease of notation, we index this grid with i for wealth and j for time. Therefore, the grid is denoted as a set $\{W_{ij}\}$. The grid defines the "state space" of the problem.

The probability of achieving the goal wealth H is the "value function" of the problem and is defined on the grid points in the state space, i.e., denoted as a set $\{V_{ij}\}$. Since the value function is also a probability, it is bounded at all points in the state space in the range $(0, 1)$.

The actions taken are denoted as a set $\{A_{ij}\}$ over each point on the state space, where each action is the choice of a portfolio, i.e., a mean and standard deviation of return pair, denoted $[\mu_{ij}, \sigma_{ij}] \in \{\mu, \sigma\}$. The vectors μ and σ are chosen from a set of admissible portfolios that the investor may use. These pairs are presented in Table 1. Therefore, the action comprises choosing the amount to invest in calls (fraction α_c of the portfolio), puts (fraction α_p), and a proportion of $(1 - \alpha_c - \alpha_p)$ in one of the 13 portfolios in Table 1, indexed by k. The action taken is also denoted as the "control" in standard dynamic programming parlance. Therefore, the action is a chosen amount of calls, puts, and the remaining balance in one of the portfolios k.

Optimization of the goal is undertaken by backward recursion on the grid. At time T, either $W_{i,T} > H$, in which case the probability of achieving the goal is $V_{i,T} = 1$ or it does not, i.e., $W_{i,T} \leq H$ and $V_{i,T} = 0$. There is no question of optimal action at time T because the portfolio

Table 1: List of mean and standard deviation pairs representing returns based on 13 portfolios that may be chosen for the dynamic investment algorithm. We also may use at-the-money calls and puts on the index, whose mean and standard deviation of return are also shown below.

Portfolio#	Mean (μ)	Standard Deviation (σ)
1	0.01	0.01479219
2	0.02	0.02955101
3	0.03	0.04609701
4	0.04	0.05903944
5	0.05	0.07547269
6	0.07	0.10410154
7	0.08	0.11911284
8	0.09	0.13401761
9	0.10	0.14892206
10	0.11	0.16382636
11	0.12	0.17873056
12	0.13	0.19647591
13	0.14	0.20533005
Index	0.0762	0.11349763

strategy terminates at that time.

Next, we do wish to decide the optimal action at time $(T-h)$. For each node i at time $j = T-h$, we choose the action $A_{i,T-h}$ that maximizes the expected value function at $V_{i,T-h}$ at state space grid point $W_{i,T-h}$. That is, we maximize the value function at each node at time $T-h$ using the Bellman (1952) equation:

$$V_{i,T-h} = \max_{w} \sum_{u} V_{u,T} \cdot Pr\left\{ \ln\left(\frac{W_{u,T|w}}{W_{i,T-h|w}} \right) \right\}, \quad \forall i \tag{20}$$

where w is an efficient portfolio choice, u is the set of grid points in the state space at time T. The transition probability, conditional on choice of efficient portfolio w, is $Pr\left\{ \ln\left(\frac{W_{u,T|w}}{W_{i,T-h|w}} \right) \right\}$ is determined using equation (18) in Section 2.3.3 in conjunction with the probability kernel fitted using the methodology specified in Section 2.3.4.

The backward recursion from T to $T-h$ may be repeated for all periods going back in time till time $t = 0$, using the general recursion:

$$V_{i,j} = \max_{w} \sum_{u} V_{u,j+h} \cdot Pr\left\{ \ln\left(\frac{W_{u,j+h|w}}{W_{i,j|w}} \right) \right\}, \quad \forall i, \forall j = 0, h, 2h, ..., T-2h \tag{21}$$

The implementation of this algorithm is easy and has low run time. The complexity is of order of the number of nodes in the state space, i.e., $|\{W_{ij}\}|$ times the number of portfolio choices to be examined at each node. The latter in our base case example works out to be four possible choices of option components, i.e., (i) no options are used in the portfolio strategy; (ii) $\alpha_c = 10\%$ of the portfolio may be invested in calls; (iii) $\alpha_p = 10\%$ of the portfolio may be invested in puts; or (iv) 10% of the portfolio may be invested in calls and another 10% in puts. Given there are 4 ways in which we may structure the options component of the strategy and 13 ways in which we can choose the non-options component, we have 52 possible portfolios to be examined in the action space at each node. Therefore, the scale of the run time is 52 times the size of the state space grid.

After backward recursion via equation (21) is complete, the node $V_{0,0}$ in the grid contains the optimized probability of reaching the goal. The corresponding action $A_{0,0}$ tells us which of the 52 portfolio choices we will begin the trading strategy with at the outset.

3 Analysis and Insights

In this section, we explore the potential improvement from using index call and put options in addition to using standard efficient portfolios. Since this allows more degrees of freedom in portfolio choice, we have to do at least as well, if not better, in maximizing the probability of reaching investor goals. This enables an examination of whether a material improvement is possible via the use of call and put options.

We begin with the following baseline case. An initial wealth of $W(0) = \$100$ is invested with a target goal of $H = \$200$ at a horizon of $T = 10$ years. As mentioned earlier the optimization problem aims to maximize the probability of reaching goal H. We also examine the probability of falling below a lower floor threshold of $L = \$100$. We then report the mean and standard deviation of the distribution of optimal terminal wealth $W(T)$.

The input data for the problem is an efficient frontier comprised of 13 portfolios in order of increasing risk and return. At any point in time the wealth in the portfolio is invested as follows: proportion α_c in calls and α_p in puts. The remaining amount $(1 - \alpha_c - \alpha_p)$ is invested in one of the efficient portfolios.

There are four cases we explore for the base case. (i) no options are used in the portfolio strategy; (ii) 10% of the portfolio may be invested in calls; (iii) 10% of the portfolio may be invested in puts; or (iv) 10% of the portfolio may be invested in calls and another 10% in puts. We then examine how these options vary in results for the base case.

3.1 Using options in the base case

The results for the four possible models in the base case are shown in the Table 2. First, from a comparison of case (1) versus the other cases, especially cases (2) and (4), we see that using options improves the outcomes. Second, the improvement comes from using calls, not puts. Third, the probability of exceeding the threshold H rises by around 8%, though the probability of exceeding the lower threshold L remains unchanged. The reason for this is that $H = 200$ is an aggressive upper threshold and call options are especially good instruments to target this goal. On the other hand the lower threshold is easily achieved and therefore can be attained without options. Therefore, there is little change in the probability of staying above the floor

Table 2: Comparison of portfolio outcomes in four cases: (i) no options are used in the portfolio strategy; (ii) $\alpha_c = 10\%$ of the portfolio may be invested in calls; (iii) $\alpha_p = 10\%$ of the portfolio may be invested in puts; or (iv) 10% of the portfolio may be invested in calls and another 10% in puts. The base case parameters are: initial wealth $W(0) = 100$; goal threshold $H = 200$; loss threshold $L = 100$; portfolio horizon $T = 10$.

	Cases			
Parameters	(1)	(2)	(3)	(4)
α_c	0.00	0.10	0.00	0.10
α_p	0.00	0.00	0.10	0.10
$Pr[W(T) \geq H]$	0.805	0.885	0.811	0.877
$Pr[W(T) \geq L]$	0.957	0.960	0.960	0.955
Mean $W(T)$	210.78	223.42	212.33	220.80
Stdev $W(T)$	45.91	50.17	46.65	49.23

even if options are used. This also explains why for this case, call options are more useful than put options.

We note that the expected wealth when options are used is higher than the base case, but it comes with additional variance as well, as is only to be expected when levered instruments like options are used. We see also that when both calls and puts are allowed, the outcomes (case 4) are very slightly lower than in case (2). This is because of the kernel density approximation, which is attenuated at the edges of the domain of the wealth distribution to a greater extent when both calls and puts are applied.

3.2 Assessing different goals

We examine how the use of options changes as the goals change, i.e., as we vary thresholds H and L. For parsimony, we only consider cases (1) and (4) and use easily achievable lower bounds, i.e., calls are more important than puts. Results are shown in Table 3. As we can see when the goal becomes more aggressive as we move H higher, the use of options becomes much more important. When the goal is only $H = 150$, the improvement in the probability of reaching this goal when options are used is about 3%. But when $H = 250$ the improvement in goal probability is four times as much, i.e., 12%. (Likewise, the standard deviation of terminal wealth is also almost three dollars higher, as is appropriate, for there can be no free lunch.) For completeness, the goal probability $Pr[W(T) \geq H]$ is shown in Figure 4. We see clearly how call options make the most difference.

3.3 The effect of approximating the true distribution

The joint distribution of one of the 13 portfolios shown in Table 1 along with the distribution of payoffs from the options on the index is approximated by the scheme presented in Section 2.3.4.

Table 3: Comparison of portfolio outcomes in two cases: (i) Case (1): no options are used in the portfolio strategy; (ii) Case (4) $\alpha_c = 10\%$ of the portfolio may be invested in calls and another $\alpha_p = 10\%$ in puts. The base parameters are: initial wealth $W(0) = 100$; portfolio horizon $T = 10$. All other parameters are shown in the table below.

	Goals				
Parameters	$H = 150$ $L = 80$	$H = 175$ $L = 90$	$H = 200$ $L = 100$	$H = 225$ $L = 110$	$H = 250$ $L = 120$
$Pr[W(T) \geq H]$					
Case (1):	0.917	0.870	0.805	0.751	0.691
Case (4):	0.948	0.918	0.877	0.848	0.814
$Pr[W(T) \geq L]$					
Case (1):	0.983	0.973	0.957	0.946	0.930
Case (4):	0.982	0.972	0.955	0.946	0.934
Mean $W(T)$					
Case (1):	169.54	190.16	210.78	229.83	247.32
Case (4):	175.20	197.20	220.80	242.95	267.03
Stdev $W(T)$					
Case (1):	25.78	34.36	45.91	58.21	70.26
Case (4):	27.89	37.52	49.23	59.87	72.30

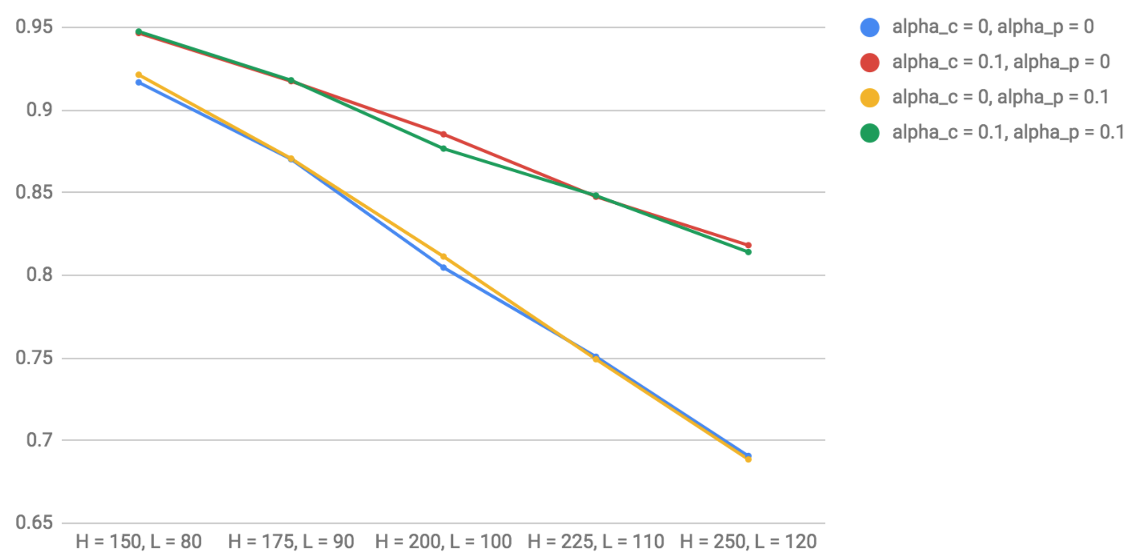

Figure 4: Comparison of goal probability $Pr[W(T) \geq H]$ in four cases: (i) no options are used in the portfolio strategy; (ii) $\alpha_c = 10\%$ of the portfolio may be invested in calls; (iii) $\alpha_p = 10\%$ of the portfolio may be invested in puts; or (iv) 10% of the portfolio may be invested in calls and another 10% in puts. The base case parameters are: initial wealth $W(0) = 100$; goal portfolio horizon $T = 10$. The goal and loss thresholds are varied and depicted in the graph on the x-axis.

Theoretically, the process followed by the joint process of (i) returns on one of the portfolios and (ii) returns on the index are assumed to be bivariate normal in this paper, and these returns are weighted and projected onto the univariate return distribution for wealth using the kernel density estimator (KDE) shown in Figure 2. Because the KDE only approximates the true joint distribution, the solution to the dynamic program will perforce be inferior to a situation when the transition probabilities are analytical. The question is, how much attenuation in accuracy is experienced when using the KDE approximation? The KDE does not have an infinite domain and since it is truncated there is some displacement of probability density versus the true analytical distribution.

Since we do not have the true transition probability density when options are used, we instead compare the performance of our algorithm when no options are used to get a baseline error from the numerical KDE approximation. For this case, we are able to use the true analytical transition probability function in backward recursion equation (21). Table 4 displays the results. The difference in optimized probability ranges from 2-4% and increases as the goals become more aggressive. Therefore, the KDE-based algorithm performs very well and is a useful way to capture complex projections of multivariate distributions in an optimization context.

3.4 The usage of calls and puts

It is of interest to examine in which states $[W(t), t]$ on the wealth grid options are included in the portfolio, and when they are not.

Table 4: Comparison of portfolio outcomes when the true analytical distribution is used versus the numerical approximation from the KDE. We coded a corresponding dynamic program for the analytical case. This is done for the case where no options are used in the portfolio strategy. The base parameters are: initial wealth $W(0) = 100$; portfolio horizon $T = 10$. All other parameters are shown in the table below. We report the percentage improvement relative to the KDE estimator.

	Goals				
	$H = 150$	$H = 175$	$H = 200$	$H = 225$	$H = 250$
Parameters	$L = 80$	$L = 90$	$L = 100$	$L = 110$	$L = 120$
$Pr[W(T) \geq H]$					
Analytical:	0.933	0.890	0.835	0.781	0.721
% improvement vs KDE:	1.79	2.31	3.73	4.08	4.38
$Pr[W(T) \geq L]$					
Analytical:	0.988	0.979	0.967	0.956	0.943
% improvement vs KDE:	0.46	0.58	1.11	1.09	1.31
Mean $W(T)$					
Analytical:	171.30	192.27	214.73	233.27	251.99
% improvement vs KDE:	1.04	1.11	1.88	1.50	1.89
Stdev $W(T)$					
Analytical:	23.31	31.86	42.91	53.49	65.65
% improvement vs KDE:	-9.55	-7.29	-6.53	-8.10	-6.56

We note that calls are used extensively but puts are not. Intuitively, calls help reach the goal, especially when the hurdle is high, whereas puts do not. It is well known from option pricing theory, see Coval and Shumway (2001), that under the risk-neutral pricing measure, both calls and puts have an expected return equal to the risk free rate. However, for the portfolio the expected return is taken under the physical probability measure, where the index is assumed to grow at its expected return, which is greater than the risk free rate. As a consequence, the drift of the index under the physical probability measure is greater than the risk free rate, making the return on calls positive and greater than that of the risk free rate, but making the return on puts correspondingly less than the risk free rate and usually negative. Therefore, since the expected return on calls is positive and that on puts is negative, puts are not an entirely sound investment unless they offset another risk that cannot be met by holding any other sort of security, such as a floor requirement on the portfolio.

On another note, many financial advisors use puts to hedge their clients' portfolios. Our analysis shows that this is unnecessary in most cases and that using a judicious mix of assets and options will also deliver a high floor on wealth while reaching optimally for goals.

Therefore, we extended the proportions of the portfolio that we might invest in calls to the following proportions: $\{0, 0.1, 0.2, 0.3\}$. There is a wide range of cases in which we use all levels of calls to improve the probability of meeting the investor's goal. The performance improvement is non-trivial as we see in Table 5. For instance, note that at low levels of goals, where $H = 150$, the improvement in the probability of reaching the goal is about 4.5% (from 0.917 to 0.959). However, when the goal is far more aggressive ($H = 250$), then without calls the probability of reaching the goal is only 0.691, whereas the goal probability is 0.864 when calls are allowed, i.e., an improvement of 17%, which is substantial. The benefit of using options, especially for investors with high goals is clear. The expected final wealth is higher when calls are used, as we see for all the five levels of goals in Table 5. At a goal level of $H = 150$, the percentage improvement in mean wealth is about 5% whereas at a goal level of $H = 250$, the improvement is about 11%, much higher, as expected. Of course, these gains from options do not come for free, investors ends up with portfolios that have higher risk, as the standard deviation also increases in lockstep.

In order to see when calls are used more often we also plotted the state space to show what proportion of the portfolio is held in calls, see Figure 5. In this figure, we can see that calls are used when the portfolio is below its initial level and tend to get used more as the portfolio grows. However, when the portfolio does poorly, we see that very few calls are used. Most of the time, we let calls go all the way to 30% of the portfolio. Therefore, either no calls are used, but if they are, then we tend to use the highest allowable levels of calls.

3.5 Using mostly options

We also examined a mostly pure options portfolio by using the following choices for the proportion of options in the portfolio: $\alpha_c = \{0.6, 0.7, 0.8, 0.9\}$. These choices imply extremely high levels of portfolio leverage, often as much as 10x. This leads naturally to much higher returns, but also much higher standard deviation. Table 6 compares this new case with the case where the options proportion is in the set $\alpha_c = \{0, 0.1, 0.2, 0.3\}$. It is clear from the table that using mostly

Table 5: Comparison of portfolio outcomes in the case where the proportion in calls ranges over $\{0, 0.1, 0.2, 0.3\}$. The base parameters are: initial wealth $W(0) = 100$; portfolio horizon $T = 10$. All other parameters are shown in the table below. We consider the cases with no calls and compare it to the case when calls are used.

	Goals				
	$H = 150$	$H = 175$	$H = 200$	$H = 225$	$H = 250$
Parameters	$L = 80$	$L = 90$	$L = 100$	$L = 110$	$L = 120$
$Pr[W(T) \geq H]$					
No calls:	0.917	0.870	0.805	0.751	0.691
With calls:	0.959	0.938	0.913	0.890	0.864
$Pr[W(T) \geq L]$					
No calls:	0.983	0.973	0.957	0.946	0.930
With calls:	0.977	0.965	0.949	0.936	0.920
Mean $W(T)$					
No calls:	169.54	190.16	210.78	229.83	247.32
With calls:	177.41	200.78	227.07	250.29	274.93
Stdev $W(T)$					
No calls:	25.78	34.36	45.91	58.21	70.26
With calls:	31.63	42.27	55.63	68.52	83.44

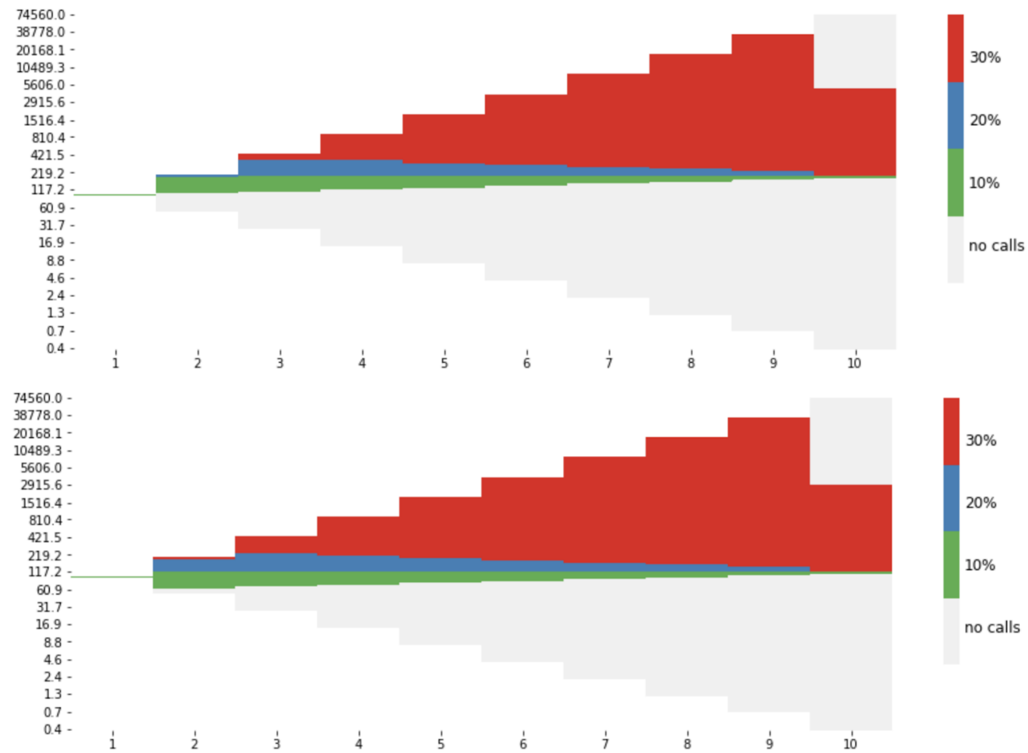

Figure 5: Extent of calls held in the portfolio as a function of the level of wealth. The proportion in calls ranges over $\{0, 0.1, 0.2, 0.3\}$. The base parameters are: initial wealth $W(0) = 100$; portfolio horizon $T = 10$. The top plot shows call holdings for a goal wealth of 150 and the lower plot shows the goal wealth for a position of 250.

Table 6: Comparison of portfolio outcomes when the proportion of options in the portfolio is: $\alpha_c = \{0.6, 0.7, 0.8, 0.9\}$ (i.e., high leverage), versus the case when we have low usage of options, i.e., $\alpha_c = \{0, 0.1, 0.2, 0.3\}$ (low leverage). The base parameters are: initial wealth $W(0) = 100$; portfolio horizon $T = 10$. All other parameters are shown in the table below. The KDE is used in both cases.

	Goals				
Parameters	$H = 150$ $L = 80$	$H = 175$ $L = 90$	$H = 200$ $L = 100$	$H = 225$ $L = 110$	$H = 250$ $L = 120$
$Pr[W(T) \geq H]$					
Low leverage:	0.959	0.938	0.913	0.890	0.864
High leverage:	0.745	0.752	0.734	0.718	0.702
$Pr[W(T) \geq L]$					
Low leverage:	0.977	0.965	0.949	0.936	0.920
High leverage:	0.817	0.828	0.815	0.804	0.794
Mean $W(T)$					
Low leverage:	177.41	200.78	227.07	250.29	274.93
High leverage:	3396	3874	3906	3931	3957
Stdev $W(T)$					
Low leverage:	31.63	42.27	55.63	68.52	83.44
High leverage:	10120	10938	10989	11029	11071

options gives extremely different results, but weaker in the sense that the goal probability drops by a material factor. This suggests that using this approach is not ideal in goals-based wealth management. Also, the probability of exceeding the lower threshold is also lower and this is not ideal. Because most of the time, the strategy maxes out the proportion of calls at 90% of the portfolio, we see that this is high enough leverage that both the mean wealth and its standard deviation explode.

It is of course interesting to allow a wide range of options (upto 0.9 of the portfolio's wealth) to see if this makes a difference and indeed, it does. See Table 7 and Figure 6.

We see a substantial increase in the probability of reaching goals, even more so for the aggressive goals than for the less aggressive ones. For example, when the goal is $H = 150$, the goal probability increases by around 4% but when the goal is $H = 250$ the increase in goal probability is 10%. Clearly, using more options offers a greater chance of hitting "reach" goals. As is also natural, the mean return is higher but so is the standard deviation of return. There is no free lunch, more return comes with more risk.

Table 7: Comparison of portfolio outcomes when the proportion of options in the portfolio is: $\alpha_c = \{0, 0.1, 0.2, 0.3, 0.5, 0.7, 0.9\}$, versus the case when we have low usage of options, i.e., $\alpha_c = \{0, 0.1, 0.2, 0.3\}$. The base parameters are: initial wealth $W(0) = 100$; portfolio horizon $T = 10$. All other parameters are shown in the table below. The KDE is used in both cases.

	Goals				
	$H = 150$	$H = 175$	$H = 200$	$H = 225$	$H = 250$
Parameters	$L = 80$	$L = 90$	$L = 100$	$L = 110$	$L = 120$
$Pr[W(T) \geq H]$					
Low leverage:	0.959	0.938	0.913	0.890	0.864
More leverage:	0.991	0.986	0.979	0.971	0.962
$Pr[W(T) \geq L]$					
Low leverage:	0.977	0.965	0.949	0.936	0.920
More leverage:	0.993	0.989	0.983	0.977	0.970
Mean $W(T)$					
Low leverage:	177.41	200.78	227.07	250.29	274.93
More leverage:	247.76	269.26	296.06	321.27	343.80
Stdev $W(T)$					
Low leverage:	31.63	42.27	55.63	68.52	83.44
More leverage:	124.94	143.71	166.07	189.92	213.27

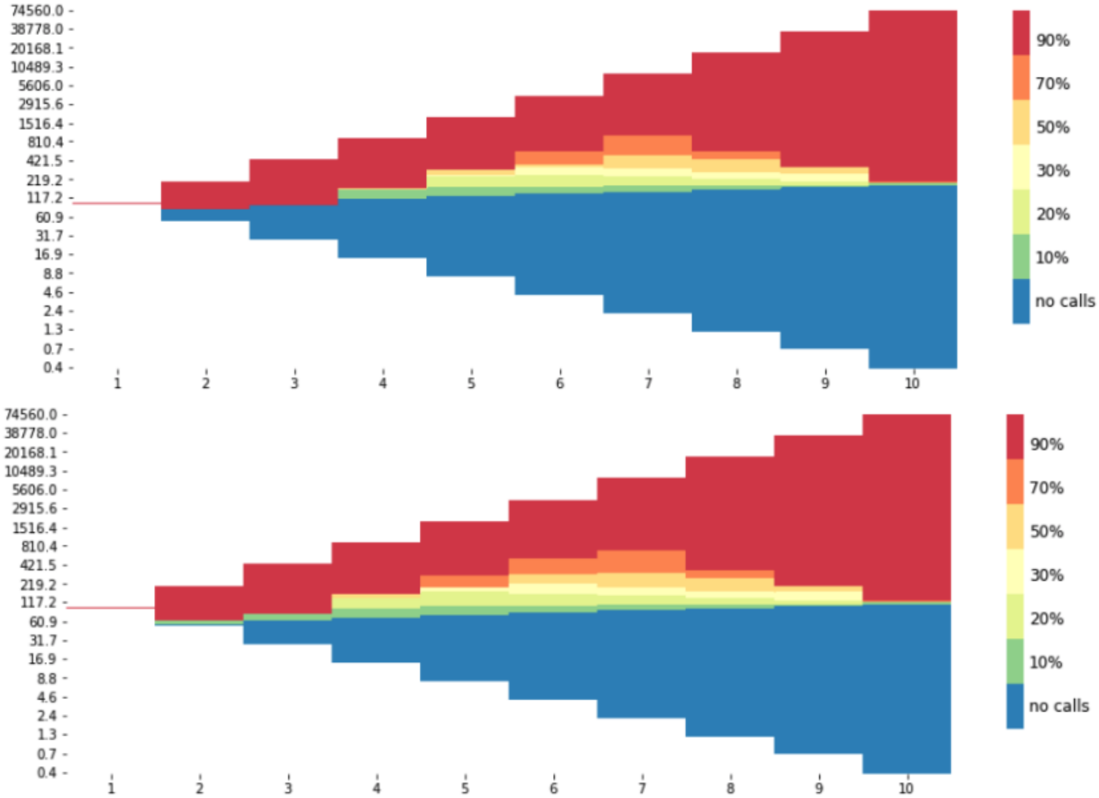

Figure 6: Extent of calls held in the portfolio as a function of the level of wealth. The proportion in calls ranges over $\{0, 0.1, 0.2, 0.3, 0.5, 0.7, 0.9\}$. The base parameters are: initial wealth $W(0) = 100$; portfolio horizon $T = 10$. The top plot shows call holdings for a goal wealth of 150 and the lower plot shows the goal wealth for a position of 250.

Table 8: Comparison of portfolio outcomes when the normal distribution is used versus the t distribution (degrees of freedom = 5) from the KDE. The base parameters are: initial wealth $W(0) = 100$; portfolio horizon $T = 10$. All other parameters are shown in the table below. The KDE is used in both cases.

	Goals				
Parameters	$H = 150$ $L = 80$	$H = 175$ $L = 90$	$H = 200$ $L = 100$	$H = 225$ $L = 110$	$H = 250$ $L = 120$
$Pr[W(T) \geq H]$					
Normal:	0.959	0.938	0.913	0.890	0.864
t-dist:	0.950	0.929	0.905	0.883	0.860
$Pr[W(T) \geq L]$					
Normal:	0.977	0.965	0.949	0.936	0.920
t-dist:	0.970	0.956	0.941	0.928	0.913
Mean $W(T)$					
Normal:	177.41	200.78	227.07	250.29	274.93
t-dist:	190.77	214.80	243.93	268.50	294.57
Stdev $W(T)$					
Normal:	31.63	42.27	55.63	68.52	83.44
t-dist:	156.15	198.15	240.13	272.15	308.46

3.6 The effect of fat-tailed distributions

The implemented strategy will change if the stochastic process is fat-tailed, as is often the case. This is easily implemented in our framework by changing the random numbers generated in step 1 of Section 2.3.4 from being normal to being drawn from a t-distribution with the required low degrees of freedom (< 10) to be fat-tailed. A comparison of results between the normal and the t-distribution is shown in Table 8.

The results for fat-tailed case are interesting. We see a small decrease in the probability of reaching goals, by around 1%. The same is noticed for the probability of exceeding the lower threshold. This is because the distribution entails more risk and the model is not optimized with reference to the lower bound. However, there is a material increase in the mean terminal wealth, offset by a substantial increase in standard deviation, which is expected given the tails of the distribution are much fatter than that of the Gaussian.

3.7 Including more complex options and structured products

Our approach is completely extensible and computationally feasible when including more complex derivative securities in the portfolio. As an example, we consider a volatility product, known

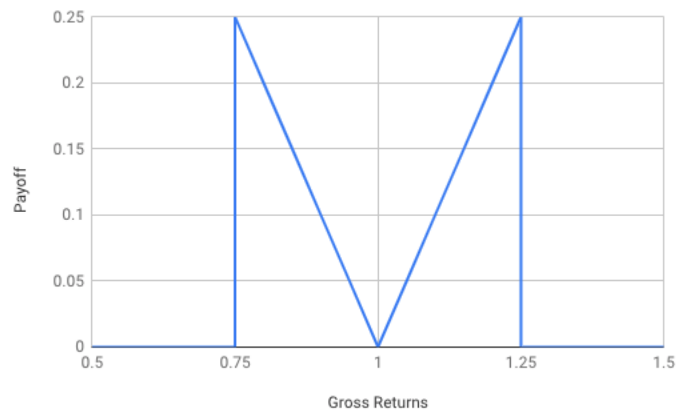

Figure 7: Graphical description of the Barrier M-note. The x-axis denotes the gross return $R(t+h) = I(t+h)/I(t)$, i.e., a return of zero means $R = 1$. Here $K = 0.25$. The payoff is $\max[0, |R - K|]$

as a "Barrier M-Note", which has a payoff that is dependent on volatility of the stock index.[2] Assuming an index value normalized to 1, a M-note pays off a return equal to $R_m(t) = |I(t) - 1|$ if $R_m(t) \leq K$, else it pays zero. For example, if $K = 0.25$, then the payoff return at maturity of the note will be 0.20 if the index reaches 1.20 or 0.80. However, if the index ends up above $(1+K)$ or below $(1-K)$ then the M-note pays nothing. Therefore, the payoff profile looks like a truncated straddle. By truncation, the seller of the note keeps the price of the truncated straddle affordable. A depiction of the payoff profile of the M-note is shown in Figure 7.

The M-note was analyzed in Das and Statman (2013) where it is shown that the note can be decomposed into 6 simpler options, which are as follows:

1. A long call at strike 1.
2. A long put at strike 1.
3. A short call at strike $1 + K$.
4. A short put at strike $1 - K$.
5. K short cash-or-nothing unit payoff calls at strike $1 + K$.
6. K short cash-or-nothing unit payoff puts at strike $1 - K$.

We have seen the pricing equations for calls and puts earlier in the paper. The price of a unit payoff cash-or-nothing option pays off $1 if the option ends up in the money. The cash-or-nothing unit payoff call price is as follows:

$$C^{(cn)} = e^{-rh} N(d_2^c); \quad d_1^c = \frac{\ln\left(\frac{1}{1+K}\right) + (r + \frac{1}{2}\sigma_I^2)h}{\sigma_I \sqrt{h}}; \quad d_2^c = d_1^c - \sigma\sqrt{h} \tag{22}$$

[2]Indexes are usually used for the underlying so as to minimize the probability of manipulation of the market in which these indexes trade.

And the corresponding put price is:

$$P^{(cn)} = e^{-rh}N(-d_2^p); \quad d_1^p = \frac{\ln\left(\frac{1}{1-K}\right) + (r + \frac{1}{2}\sigma_I^2)h}{\sigma_I\sqrt{h}}; \quad d_2^p = d_1^p - \sigma\sqrt{h} \tag{23}$$

Using these equations, we can price the M-note at time t, denoted $M(t)$ and the return on the M-note is the payoff divided by the price, i.e., $\max[0, |R(t+h) - K|]/M(t)$.

Now we are ready to extend the model in Section 2.3.3 to include the M-note as an option in the portfolio. Let $\alpha_m(t)$ be the proportion of wealth $W(t)$ invested in the M-note. The number of units of the M-note will be:

$$n_m(t) = \frac{\alpha_m(t) \cdot W(t)}{M(t)} \tag{24}$$

The net wealth invested in the equity portfolio is

$$W'(t) = W(t)[1 - \alpha_c(t) - \alpha_p(t) - \alpha_m(t)] \tag{25}$$

which corresponds to equation (14) from earlier. Total wealth will evolve as follows:

$$\begin{aligned} W(t+h) &= W'(t)\exp\left[\left(\mu(t) - \frac{1}{2}\sigma(t)^2\right)h + \sigma(t)\sqrt{h}\cdot Z(t)\right] \\ &\quad + n_c(t)\max[0, I(t+h) - I(t)] \\ &\quad + n_p\max[0, I(t) - I(t+h)] \\ &\quad + n_m(t)\max[0, |I(t+h)/I(t) - K|] \end{aligned} \tag{26}$$

Equation (18) is then extended to the following:

$$\begin{aligned} \frac{W(t+h)}{W(t)} &= [1 - \alpha_c - \alpha_p - \alpha_m]\exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)h + \sigma\sqrt{h}\cdot Z\right] \\ &+ \frac{\alpha_c(t)}{[N(d_1) - e^{-rh}N(d_2)]}\max\left\{0, \exp\left[\left(\mu_I - \frac{1}{2}\sigma_I^2\right)h + \sigma_I\sqrt{h}\cdot Z_I\right] - 1\right\} \\ &+ \frac{\alpha_p(t)}{[e^{-rh}N(-d_2) - N(-d_1)]}\max\left\{0, 1 - \exp\left[\left(\mu_I - \frac{1}{2}\sigma_I^2\right)h + \sigma_I\sqrt{h}\cdot Z_I\right]\right\} \\ &+ \frac{\alpha_m(t)}{M(t)}\cdot M(t+h) \end{aligned} \tag{27}$$

where

$$M(t+h) = \left| \exp\left[\left(\mu_I - \frac{1}{2}\sigma_I^2\right)h + \sigma_I\sqrt{h}\cdot Z_I\right] - 1 \right| \tag{28}$$

if $M(t+h) \leq K$, else $M(t+h) = 0$. (Note that the last term contains the absolute sign function.) The same fast kernel density estimator may be applied using the simulated values of $\{Z, Z_I\}$ in equation (27). Results are in Table 9. The probability of reaching the goal is improved with a much lower risk strategy as well, because less leverage is adopted.

Table 9: Comparison of portfolio outcomes when the proportion of options in the portfolio is: $\alpha_c = \{0, 0.1, 0.2, 0.3, 0.5, 0.7, 0.9\}$, versus the case when we have all these options plus the M-note with $K = 0.25$. The base parameters are: initial wealth $W(0) = 100$; portfolio horizon $T = 10$. All other parameters are shown in the table below. The KDE is used in both cases.

	Goals				
	$H = 150$	$H = 175$	$H = 200$	$H = 225$	$H = 250$
Parameters	$L = 80$	$L = 90$	$L = 100$	$L = 110$	$L = 120$
$Pr[W(T) \geq H]$					
M-note & Calls:	0.998	0.997	0.996	0.995	0.993
Calls only:	0.991	0.986	0.979	0.971	0.962
$Pr[W(T) \geq L]$					
M-note & Calls:	0.999	0.999	0.998	0.997	0.996
Calls only:	0.993	0.989	0.983	0.977	0.970
Mean $W(T)$					
M-note & Calls:	182.38	217.69	253.34	279.96	308.89
Calls only:	247.76	269.26	296.06	321.27	343.80
Stdev $W(T)$					
M-note & Calls:	33.84	62.60	83.07	95.00	113.04
Calls only:	124.94	143.71	166.07	189.92	213.27

4 Concluding Discussion

In this paper we develop a dynamic programming solution for goal-based wealth management when options are included in the trading strategy. This involves a sharp increase in both, the dimensionality of the problem's state space and also highly non-Gaussian distributions. We develop a facile mathematical approach to address both these issues using kernel density estimators. This approach is also computationally efficient.

Using this approach we find that portfolio outcomes, especially for long-horizon portfolios, are much improved when options use is permitted. We explore the use of calls versus puts, and find that the former are mostly used, and puts are unnecessary unless floor constraints are paramount. The use of calls makes it much more likely that an investor will achieve aggressive wealth management goals. We also find that pure options strategies are not sufficient, and derivatives need to be mixed with equities and bonds (standard mean-variance portfolios) to get best results. The optimal dynamic strategy also varies widely in how much of the portfolio is invested in options, depending on the state of the portfolio relative to its goals. The methodology is easily extended to including structured products and volatility derivatives and an example using an M-note is also presented. The results in this paper strongly advocate for the use of options in dynamic goals-based wealth management.

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